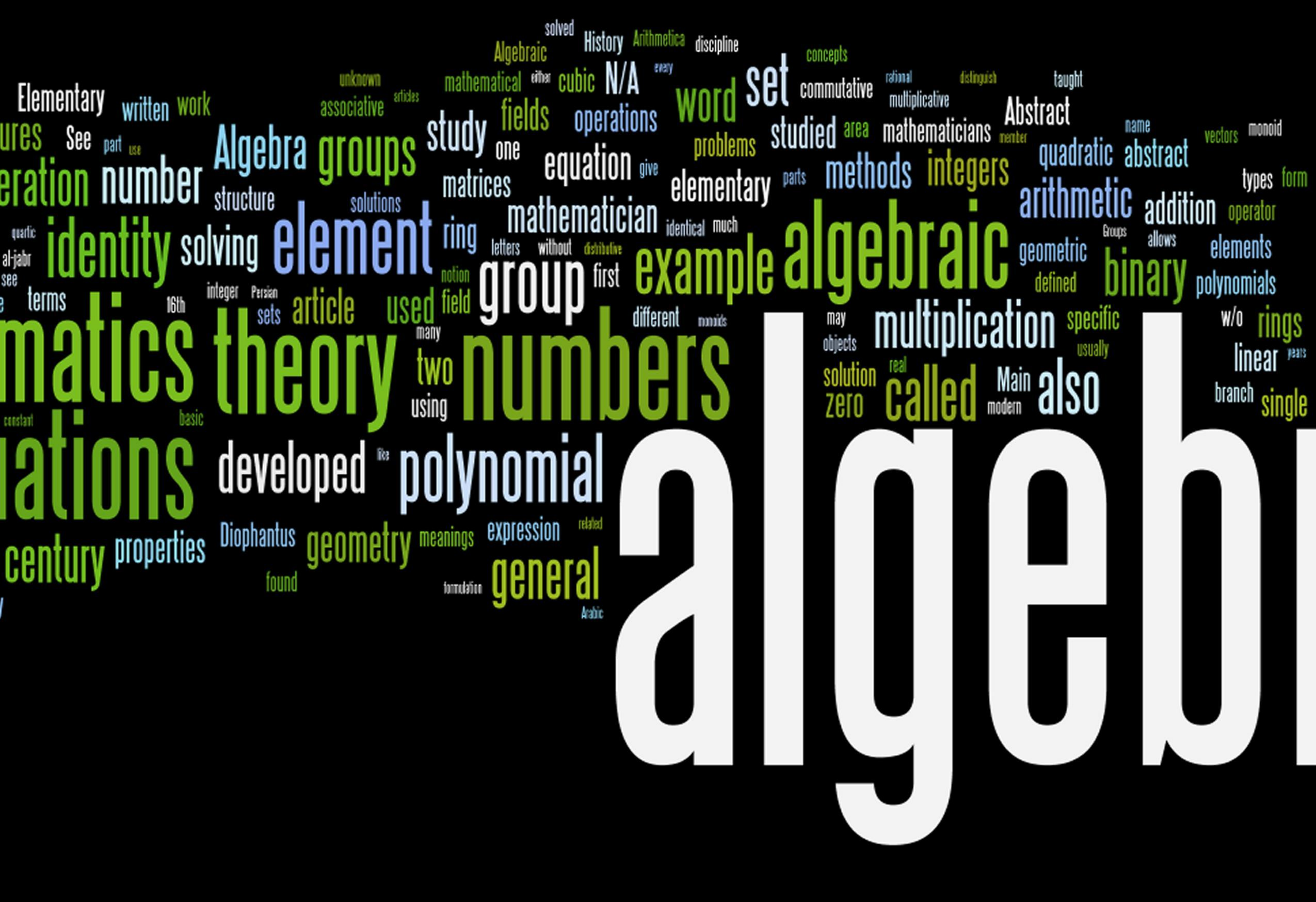
CK-12

flexbook
next generation textbooks

ABSE Math 4



ABSE Math 4

Julie Pfaff

AUTHOR
Julie Pfaff

To access a customizable version of this book, as well as other interactive content, visit www.ck12.org

CK-12 Foundation is a non-profit organization with a mission to reduce the cost of textbook materials for the K-12 market both in the U.S. and worldwide. Using an open-content, web-based collaborative model termed the **FlexBook®**, CK-12 intends to pioneer the generation and distribution of high-quality educational content that will serve both as core text as well as provide an adaptive environment for learning, powered through the **FlexBook Platform®**.

Printed: April 25, 2014

flexbook
next generation textbooks

Contents

CHAPTER 1 Chapter 1: Real Numbers

Chapter Outline

1.1 The Real Number System

Here you'll learn how to identify the subsets of real numbers and place a real number into one of these subsets.

There are many different ways to classify or name numbers.

All of the numbers we'll work with in this class are considered *real numbers*.

When you were in the lower grades, you worked with ***whole numbers*. Whole numbers are counting numbers. We consider whole numbers as the set of numbers** $\{0,1,2,3,4\dots\}$.

Later, you may also have learned about ***integers***. **The set of integers includes whole numbers, but also includes their opposites.** Therefore, we can say that whole positive and negative numbers are part of the set of integers $\{\dots -2,-1,0,1,2,3\dots\}$.

We can't stop classifying numbers with whole numbers and integers because sometimes we can measure a part of a whole or a whole with parts. These numbers are called ***rational numbers*. A rational number is any number that can be written as a fraction where the numerator or the denominator is not equal to zero.** Let's think about this. A whole number or an integer could also be a rational number because we can put it over 1.

-4 could be written as $-\frac{4}{1}$, therefore it is an integer, but also a rational number.

Exactly. We can also think about decimals too. Many decimals can be written as fractions, so decimals are also rational numbers.

There are two special types of decimals that are considered rational numbers and one kind of decimal that is NOT a rational number. A ***terminating decimal*** is a decimal that is considered to be a rational number. **A terminating decimal is a decimal that looks like it goes on and on, but at some point has an end.** It terminates or ends somewhere.

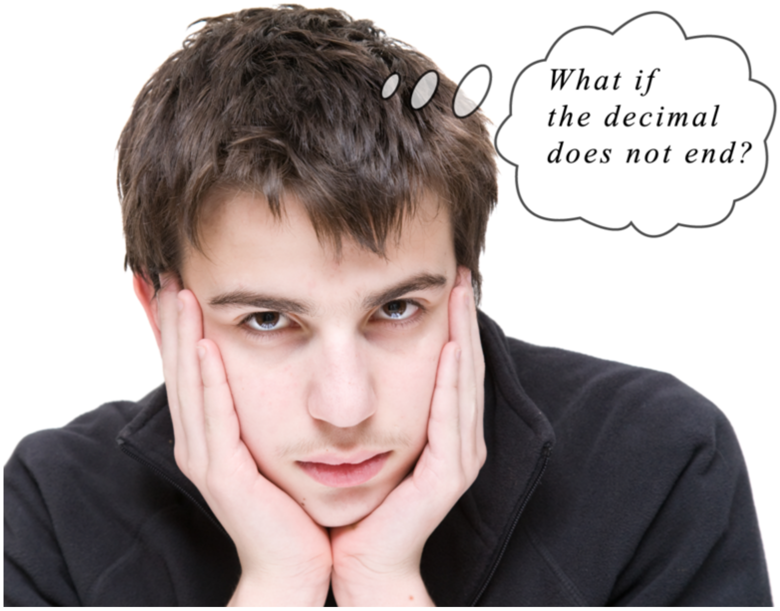
.3456798

This is a terminating decimal. It goes on for a while, but then ends.

A ***repeating decimal*** is also considered a rational number. **A repeating decimal has values that repeat forever.**

.676767679...

This is a repeating decimal.



Ah ha! This is the last type of number that is a decimal, but is NOT a rational number. It is called an *irrational number.* An irrational number is a decimal that does not end and has no repetition. It goes on and on and on. Irrational numbers cannot be represented as fractions. The most famous irrational number is *pi* (π). We use 3.14 to represent π, but you should know that *pi* is an irrational number meaning that it goes on and on and on forever.

How can we determine if a fraction or a decimal is rational or irrational?

If a number can be written in fraction form then it is rational. If a number cannot be written in fraction form then it is irrational. Besides π, roots of many numbers are also examples of irrational numbers. For example, $\sqrt{2}$ and $\sqrt{3}$ are both irrational numbers.

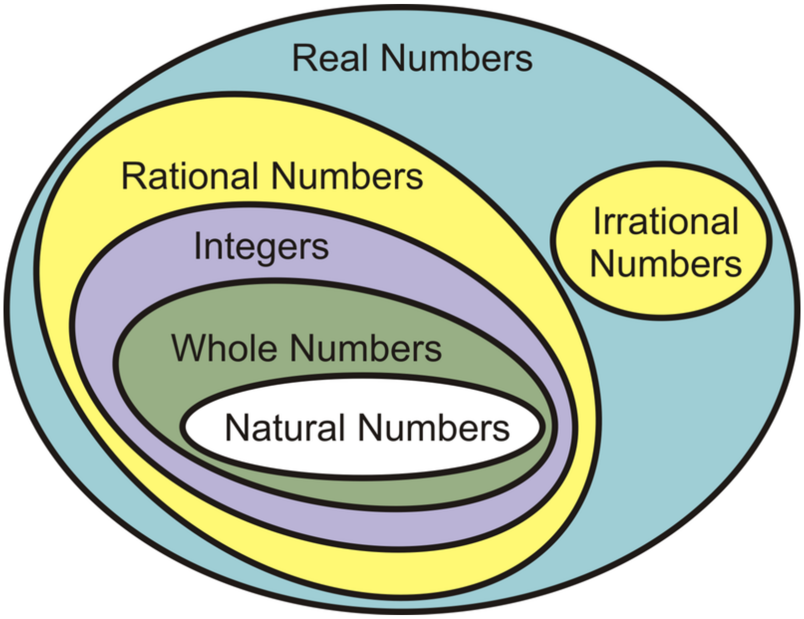
The table and diagram below summarize the real number system:

TABLE 1.1:

Real Numbers	Any number that can be plotted on a number line. *Symbol:* $\mathbb{R}$	*Examples:*$8, 4.67, -\frac{1}{3}, \pi$
Rational Numbers	Any number that can be written as a fraction, including repeating decimals. *Symbol:* $\mathbb{Q}$	*Examples:*$-\frac{5}{9}, \frac{1}{8}, 1.\overline{3}, \frac{16}{4}$
Irrational Numbers	Real numbers that are not rational. When written as a decimal, these numbers do not end nor repeat.	*Example:*$e, \pi, -\sqrt{2}, \sqrt[3]{5}$
Integers	All positive and negative "counting" numbers and zero. *Symbol:* $\mathbb{Z}$	*Example:* -4, 6, 23, -10
Whole Numbers	All positive "counting" numbers and zero.	*Example:* 0, 1, 2, 3, ...
Natural Numbers	All positive "counting" numbers. *Symbol:* $\mathbb{N}$	*Example:* 1, 2, 3, ...

A **counting number** is any number that can be counted on your fingers.

The real numbers can be grouped together as follows:



Classifying Real Numbers

Classify each real number.

Example A

$\sqrt{7}$

Solution: Irrational number

Example B

$\frac{1}{9}$

Solution: Rational number

Example C

-98

Solution: Integer and rational number

Writing a Repeating Decimal as a Fraction

Example: How do we write 0.14141414.... as a fraction? Let's devise a step-by-step process.

Step 1: Set your repeating decimal equal to x.

$$x = 0.14141414$$

Step 2: Find the repeating digit(s).

In this case 14 is repeating.

Step 3: Move the repeating digits to the left of the decimal point and leave the remaining digits to the right.

$$14.14141414$$

Step 4: Multiply x by the same factor you mulitplied your original repeating decimal to get your new repeating decimal.

$$14.14141414 = 100(0.14141414)$$

So,

$$100x = 14.14141414$$

Step 5: Solve your system of linear equations for x.

$$(100x = 14.14141414) - (x = 0.14141414)$$

yields:

$$99x = 14$$

, so $x = \frac{14}{99}$

What about 0.327272727... ? The 0.3 does not repeat. So, rewrite this as 0.727272727... – 0.4 Therefore, the fraction will be:

$$\begin{aligned} \frac{72}{99} &- \frac{4}{10} \\ \frac{8}{11} &- \frac{2}{5} \\ \frac{40}{55} &- \frac{22}{55} \\ &\frac{18}{55} \end{aligned}$$

Vocabulary

Subset
A set of numbers that is contained in a larger group of numbers.

Real Numbers
Any number that can be plotted on a number line.

Rational Numbers
Any number that can be written as a fraction, including repeating decimals.

Irrational Numbers
Real numbers that are not rational. When written as a decimal, these numbers do not end nor repeat.

Integers
All positive and negative "counting" numbers and zero.

Whole Numbers
All positive "counting" numbers and zero.

Natural Numbers or Counting Numbers
Numbers than can be counted on your fingers; 1, 2, 3, 4, ...

Terminating Decimal
When a decimal number ends.

Repeating Decimal
When a decimal number repeats itself in a pattern. 1.666..., 0.98989898... are examples of repeating decimals.

Guided Practice

1. What type of real number is $\sqrt{5}$?

2. List all the subsets that -8 is a part of.

3. True or False: $-\sqrt{9}$ is an irrational number.

Answers

1. $\sqrt{5}$ is an irrational number because, when converted to a decimal, it does not end nor does it repeat.

2. -8 is a negative integer. Therefore, it is also a rational number and a real number.

3. $-\sqrt{9}=-3$, which is an integer. The statement is false.

Practice

Directions: Classify each of the following numbers as real, whole, integer, rational or irrational. Some numbers will have more than one classification.

1. 3.45
2. -9
3. 1,270
4. 1.232323
5. $\frac{4}{5}$
6. -232,323
7. -98
8. 1.98
9. $\sqrt{16}$
10. $\sqrt{2}$

Directions: Answer each question as true or false.

11. An irrational number can also be a real number.
12. An irrational number is a real number and an integer.
13. A whole number is also an integer.
14. A decimal is considered a real number and a rational number.
15. A negative decimal can still be considered an integer.
16. An irrational number is a terminating decimal.
17. A radical is always an irrational number.
18. Negative whole numbers are integers and are also rational numbers.
19. Pi is an example of an irrational number.
20. A repeating decimal is also a rational number.

Rewrite the following repeating decimals as fractions.

21. 0.4646464646...
22. 0.81212121212...

1.2 Integer Operations

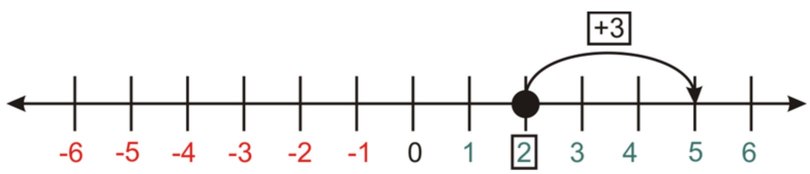
Here you'll learn the Commutative Property of Addition, Associative Property of Addition, and Identity Property of Addition so that you can effectively add integers.

Addition of Integers

A football team gains 11 yards on one play, then loses 5 yards on the next play, and then loses 2 yards on the third play. What is the total loss or gain of yardage?

A loss can be expressed as a negative integer. A gain can be expressed as a positive integer. To find the net gain or loss, the individual values must be added together. Therefore, the sum is $11+(-5)+(-2)=4$. The team has a net gain of 4 yards.

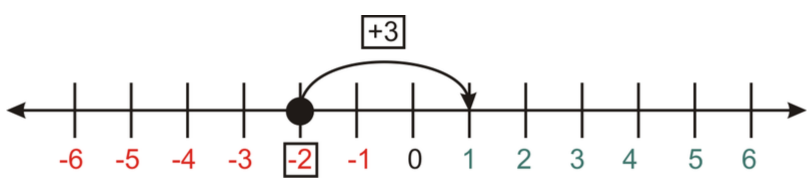
Addition can also be shown using a number line. If you need to add $2+3$, start by making a point at the value of 2 and move three integers to the right. The ending value represents the sum of the values.



Example A

Find the sum of $-2+3$ *using a number line.*

Solution: Begin by making a point at –2 and moving three units to the right. The final value is 1, so $-2+3=1$.

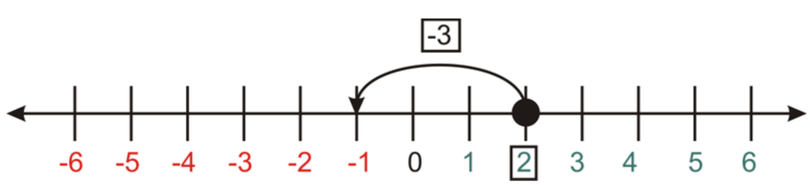


When the value that is being added is positive, we jump to the right. If the value is negative, we jump to the left (in a negative direction).

Example B

Find the sum of $2-3$ *using a number line.*

Solution: Begin by making a point at 2. The expression represents subtraction, so we will count three jumps to the left.



The solution is: $2-3=-1$.

Subtraction of Integers

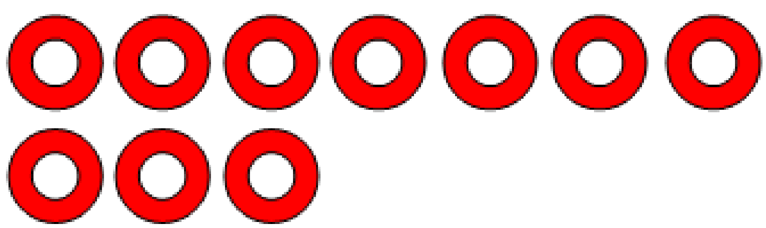
To subtract one signed number from another, change the problem from a subtraction problem to an addition problem and change the sign of the number that was originally being subtracted. In other words, to subtract signed numbers simply add the opposite. Then, follow the rules for adding signed numbers.

The subtraction of integers can be represented with manipulatives such as color counters and algebra tiles. A number line can also be used to show the subtraction of integers.

Example A

$7-(-3)=?$

Solution: This is the same as $7+(+3)=?$. The problem can be represented with color counters. In this case, the red counters represent positive numbers.



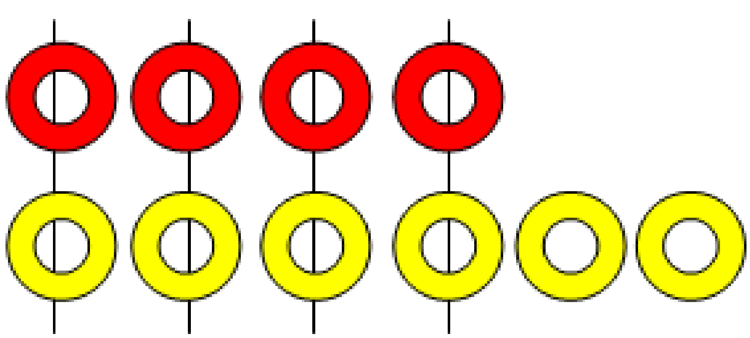
The answer is the sum of 7 and 3. $7+(+3)=10$

Example B

$4-(+6)=$

Solution: Change the problem to an addition problem and change the sign of the original number that was being subtracted.

$4-(+6)=4+(-6)=?$



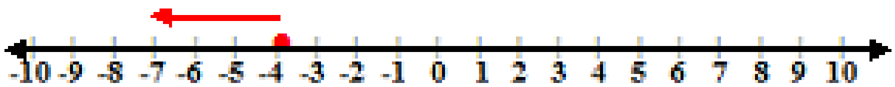
The remaining counters represent the answer. Therefore, $4-(+6)=-2$. The answer is the difference between 6 and 4 and takes the sign of the larger number.

Example C

$(-4)-(+3)=?$

Solution: This is the same as $(-4)+(-3)=?$. The solution to this problem can be determined by using the number line.

Indicate the starting point of -4 by using a dot. From this point, add a -3 by moving three places to the left. You will stop at -7.



The point where you stopped is the answer to the problem. Therefore, $(-4)-(+3)=-7$

Multiplication and Division of Integers

When you multiply and divide integers, there are some rules that need to be committed to memory.

- When multiplying or dividing a positive integer by a positive integer, the product or quotient is positive.
- When multiplying or dividing a positive integer by a negative integer, or a negative integer by a positive integer, the product or quotient is negative.
- When multiplying or dividing a negative integer by a negative integer, the product or quotient is positive.

You can remember these rules in a quicker way: if the signs are the same, the answer is positive. If the signs are different, the answer is negative.

Now let's apply these rules.

$12(-5)$

Notice that this is a multiplication problem. We use the parentheses around a single value to show multiplication. Now we can multiply the two values and then add the sign.

$12\times 5=60$

A negative value times a positive value is a negative value.

Our answer is -60.

Here is another one.

$\frac{-150}{-50}$

Notice that this is a division problem. We use the fraction bar to show division. We do the division itself first.

$150\div 50=3$

A negative divided by a negative is a positive.

The answer is 3.

Example A

$-9(7)$

Solution: -63

Example B

$-3(-12)$

Solution: 36

Example C

$-169 \div 13$

Solution: -13

Practice

Subtract.

1. $(-9)-(-2)$
2. $(5)-(+8)$
3. $(5)-(-4)$
4. $(-7)-(-9)$
5. $(6)-(+5)$
6. $(8)-(+4)$
7. $(-2)-(-7)$
8. $(3)-(+5)$
9. $(-6)-(-10)$
10. $(-4)-(-7)$
11. $(-13)-(-19)$
12. $(-6)-(+8)-(-12)$
13. $(14)-(+8)-(-6)$
14. $(18)-(+8)-(+3)$
15. $(10)-(-6)-(+4)-(+2)$

Directions: Multiply the following integers.

1. $-6(-8) =$ _____
2. $5(-10) =$ _____
3. $3(-4) =$ _____
4. $-3(4) =$ _____
5. $8(-9) =$ _____
6. $-9(12) =$ _____
7. $8(-11) =$ _____
8. $(-5)(-9) =$ _____
9. $-7(-8) =$ _____
10. $(-12)(12) =$ _____

Directions: Divide the following integers.

11. $-12 \div 2 =$ _____
12. $-18 \div -6 =$ _____
13. $-24 \div 12 =$ _____
14. $-80 \div -4 =$ _____
15. $-60 \div -30 =$ _____
16. $\frac{28}{4} =$ _____
17. $\frac{-36}{4} =$ _____
18. $\frac{-45}{-9} =$ _____
19. $-75 \div 25 =$ _____
20. $-68 \div -2 =$ _____

1.3 Order of Operations

Introduction

Look at and evaluate the following expression:

$$2+4\times 7-1=?$$

How many different ways can we interpret this problem, and how many different answers could someone possibly find for it?

The *simplest* way to evaluate the expression is simply to start at the left and work your way across:

$$\begin{aligned}&2+4\times 7-1\\&=6\times 7-1\\&=42-1\\&=41\end{aligned}$$

This is the answer you would get if you entered the expression into an ordinary calculator. But if you entered the expression into a scientific calculator or a graphing calculator you would probably get 29 as the answer.

In mathematics, the order in which we perform the various **operations** (such as adding, multiplying, etc.) is important. In the expression above, the operation of **multiplication** takes precedence over **addition**, so we evaluate it first. Let's re-write the expression, but put the multiplication in brackets to show that it is to be evaluated first.

$$2+(4\times 7)-1=?$$

First evaluate the brackets: $4\times 7=28$. Our expression becomes:

$$2+(28)-1=?$$

When we have only addition and subtraction, we start at the left and work across:

$$\begin{aligned}&2+28-1\\&=30-1\\&=29\end{aligned}$$

Algebra students often use the word **"PEMDAS"** to help remember the order in which we evaluate the mathematical expressions: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction.

Order of Operations

1. Evaluate expressions within **P**arentheses (also all brackets [ ] and braces { }) first.
2. Evaluate all **E**xponents (terms such as 3^2 or x^3) next.
3. **M**ultiplication *and* **D**ivision is next - work from left to right completing **both** multiplication and division in the order that they appear.
4. Finally, evaluate **A**ddition *and* **S**ubtraction - work from left to right completing **both** addition and subtraction in the order that they appear.

Evaluate Algebraic Expressions with Grouping Symbols

The first step in the order of operations is called **parentheses**, but we include all **grouping symbols** in this step—not just parentheses (), but also square brackets [ ] and curly braces { }.

Example 1

Evaluate the following:

a) $4-7-11+2$

b) $4-(7-11)+2$

c) $4-[7-(11+2)]$

Each of these expressions has the same numbers and the same mathematical operations, in the same order. The placement of the various grouping symbols means, however, that we must evaluate everything in a different order each time. Let's look at how we evaluate each of these examples.

a) This expression doesn't have parentheses, exponents, multiplication, or division. **PEMDAS** states that we treat addition and subtraction as they appear, starting at the left and working right (it's NOT addition *then* subtraction).

$$\begin{aligned} 4-7-11+2 &= -3-11+2 \\ &= -14+2 \\ &= -12 \end{aligned}$$

b) This expression has parentheses, so we first evaluate $7-11=-4$. Remember that when we subtract a negative it is equivalent to adding a positive:

$$\begin{aligned} 4-(7-11)+2 &= 4-(-4)+2 \\ &= 8+2 \\ &= 10 \end{aligned}$$

c) An expression can contain any number of sets of parentheses. Sometimes expressions will have sets of parentheses **inside** other sets of parentheses. When faced with **nested parentheses**, start at the innermost parentheses and work outward.

Brackets may also be used to group expressions which already contain parentheses. This expression has both brackets and parentheses. We start with the innermost group: $11+2=13$. Then we complete the operation in the brackets.

$$\begin{aligned} 4-[7-(11+2)] &= 4-[7-(13)] \\ &= 4-[-6] \\ &= 10 \end{aligned}$$

Example 2

Evaluate the following:

a) $3 \times 5 - 7 \div 2$

b) $3 \times (5-7) \div 2$

c) $(3 \times 5) - (7 \div 2)$

a) There are no grouping symbols. **PEMDAS** dictates that we multiply and divide first, working from left to right: $3 \times 5 = 15$ and $7 \div 2 = 3.5$. (NOTE: It's not multiplication *then* division.) Next we subtract:

$$\begin{aligned} 3 \times 5 - 7 \div 2 &= 15 - 3.5 \\ &= 11.5 \end{aligned}$$

b) First, we evaluate the expression inside the parentheses: $5 - 7 = -2$. Then work from left to right:

$$\begin{aligned} 3 \times (5-7) \div 2 &= 3 \times (-2) \div 2 \\ &= (-6) \div 2 \\ &= -3 \end{aligned}$$

c) First, we evaluate the expressions inside parentheses: $3 \times 5 = 15$ and $7 \div 2 = 3.5$. Then work from left to right:

$$\begin{aligned} (3 \times 5) - (7 \div 2) &= 15 - 3.5 \\ &= 11.5 \end{aligned}$$

Note that adding parentheses didn't change the expression in part c, but did make it easier to read. Parentheses can be used to change the order of operations in an expression, but they can also be used simply to make it easier to understand.

We can also use the order of operations to simplify an expression that has variables in it, after we substitute specific values for those variables.

Example 3

Use the order of operations to evaluate the following:

a) $2 - (3x + 2)$ when $x = 2$

b) $3y^2 + 2y + 1$ when $y = -3$

c) $2 - (t-7)^2 \times (u^3 - v)$ when $t = 19,\ u = 4$, and $v = 2$

a) The first step is to substitute the value for x into the expression. We can put it in parentheses to clarify the resulting expression.

$$2-(3(2)+2)$$

(Note: 3(2) is the same as 3×2.)

Follow **PEMDAS** - first parentheses. Inside parentheses follow **PEMDAS** again.

$$\begin{array}{ll} 2-(3\times 2+2)=2-(6+2) & \text{Inside the parentheses, we multiply first.} \\ 2-8=-6 & \text{Next we add inside the parentheses, and finally we subtract.} \end{array}$$

b) The first step is to substitute the value for y into the expression.

$$3\times(-3)^2+2\times(-3)-1$$

Follow **PEMDAS**: we cannot simplify the expressions in parentheses, so exponents come next.

$$\begin{array}{ll} 3\times(-3)^2+2\times(-3)-1 & \text{Evaluate exponents: } (-3)^2=9 \\ =3\times 9+2\times(-3)-1 & \text{Evaluate multiplication: } 3\times 9=27;\ 2\times -3=-6 \\ =27+(-6)-1 & \text{Add and subtract in order from left to right.} \\ =27-6-1 & \\ =20 & \end{array}$$

c) The first step is to substitute the values for t, u, and v into the expression.

$$2-(19-7)^2\times(4^3-2)$$

Follow **PEMDAS**:

$$\begin{array}{ll} 2-(19-7)^2\times(4^3-2) & \text{Evaluate parentheses: } (19-7)=12;\ (4^3-2)=(64-2)=62 \\ =2-12^2\times 62 & \text{Evaluate exponents: } 12^2=144 \\ =2-144\times 62 & \text{Multiply: } 144\times 62=8928 \\ =2-8928 & \text{Subtract.} \\ =-8926 & \end{array}$$

In parts (b) and (c) we left the parentheses around the negative numbers to clarify the problem. They did not affect the order of operations, but they did help avoid confusion when we were multiplying negative numbers.

Part (c) in the last example shows another interesting point. When we have an expression inside the parentheses, we use **PEMDAS** to determine the order in which we evaluate the contents.

Evaluate Algebraic Expressions with Fraction Bars

Fraction bars count as grouping symbols for **PEMDAS**, so we evaluate them in the first step of solving an expression. All numerators and all denominators can be treated as if they have invisible parentheses around them. When **real** parentheses are also present, remember that the innermost grouping symbols come first. If, for example, parentheses appear on a numerator, they would take precedence over the fraction bar. If the parentheses appear outside of the fraction, then the fraction bar takes precedence.

Example 4

Use the order of operations to evaluate the following expressions:

a) $\frac{z+3}{4} - 1$ when $z = 2$

b) $\left(\frac{a+2}{b+4} - 1\right) + b$ when $a = 3$ and $b = 1$

a) We substitute the value for z into the expression.

$$\frac{2+3}{4} - 1$$

Although this expression has no parentheses, the fraction bar is also a grouping symbol—it has the same effect as a set of parentheses. We can write in the "invisible parentheses" for clarity:

$$\frac{(2+3)}{4} - 1$$

Using **PEMDAS**, we first evaluate the numerator:

$$\frac{5}{4} - 1$$

We can convert $\frac{5}{4}$ to a mixed number:

$$\frac{5}{4} = 1\frac{1}{4}$$

Then evaluate the expression:

$$\frac{5}{4} - 1 = 1\frac{1}{4} - 1 = \frac{1}{4}$$

b) We substitute the values for a and b into the expression:

$$\left(\frac{3+2}{1+4} - 1\right) + 1$$

This expression has nested parentheses (remember the effect of the fraction bar). The innermost grouping symbol is provided by the fraction bar. We evaluate the numerator $(3+2)$ and denominator $(1+4)$ first.

$$\begin{aligned}\left(\frac{3+2}{1+4}-1\right)+1 &= \left(\frac{5}{5}-1\right)+1 && \text{Next we evaluate the inside of the parentheses. First we divide.}\\ &= (1-1)+1 && \text{Next we subtract.}\\ &= 0+1=1\end{aligned}$$

Additional Resources

For more practice, you can play an algebra game involving order of operations online at http://www.funbrain.com/algebra/index.html .

Review Questions

1. Use the order of operations to evaluate the following expressions.
   a. $8-(19-(2+5)-7)$
   b. $2+7\times 11-12\div 3$
   c. $(3+7)\div(7-12)$
   d. $\frac{2\cdot(3+(2-1))}{4-(6+2)}-(3-5)$
   e. $\frac{4+7(3)}{9-4}+\frac{12-3\cdot 2}{2}$
   f. $(4-1)^2+3^2\cdot 2$
   g. $\frac{(2^2+5)^2}{5^2-4^2}\div(2+1)$
2. Evaluate the following expressions involving variables.
   a. $\frac{jk}{j+k}$ when $j=6$ and $k=12$
   b. $2y^2$ when $x=1$ and $y=5$
   c. $3x^2+2x+1$ when $x=5$
   d. $(y^2-x)^2$ when $x=2$ and $y=1$
   e. $\frac{x+y^2}{y-x}$ when $x=2$ and $y=3$
3. Evaluate the following expressions involving variables.
   a. $\frac{4x}{9x^2-3x+1}$ when $x=2$
   b. $\frac{z^2}{x+y}+\frac{x^2}{x-y}$ when $x=1,\ y=-2$, and $z=4$
   c. $\frac{4xyz}{y^2-x^2}$ when $x=3,\ y=2$, and $z=5$
   d. $\frac{x^2-z^2}{xz-2x(z-x)}$ when $x=-1$ and $z=3$
4. Insert parentheses in each expression to make a true equation.
   a. $5-2\times 6-5+2=5$
   b. $12\div 4+10-3\times 3+7=11$
   c. $22-32-5\times 3-6=30$
   d. $12-8-4\times 5=-8$

1.4 Distributive Property

Here you'll learn to identify and apply the Distributive Property to evaluate numerical expressions.

Evaluating Expressions with Both Products and Sums/Differences

What does the word "evaluate" mean?

When we ***evaluate*** an expression, we figure out the value of that expression or the quantity of the expression.

When we evaluate expressions that have a product and a sum, we use a ***property*** called the Distributive Property.

What is the Distributive Property?

The Distributive Property is a property that is a true statement about how to multiply a number with a sum. Multiply the number outside the parentheses with each number inside the parentheses. Then figure out the sum of those products.

In other words, we distribute the number outside the parentheses with both of the values inside the parentheses and find the sum of those numbers.

Let's see how this works.

$4(3+2)$

To use the Distributive Property, we take the four and multiply it by both of the numbers inside the parentheses. Then we find the sum of those products.

$$\begin{aligned}&4(3)+4(2)\\&12+8\\&20\end{aligned}$$

Our answer is 20.

Here is another one.

$8(9+4)$

Multiply the eight times both of the numbers inside the parentheses. Then find the sum of the products.

$$\begin{aligned}&8(9)+8(4)\\&72+32\\&104\end{aligned}$$

Our answer is 104.

Now it is your turn. Evaluate these expressions using the Distributive Property.

Example A

$5(6+3)$

Solution: 45

Example B

$2(8+1)$

Solution: 18

Example C

$12(3+2)$

Solution: 60

Vocabulary

Numerical expression
a number sentence that has at least two different operations in it.

Product
the answer in a multiplication problem

Sum
the answer in an addition problem

Property
a rule that works for all numbers

Evaluate
to find the quantity of values in an expression

The Distributive Property
the property that involves taking the product of the sum of two numbers. Take the number outside the parentheses and multiply it by each term in the parentheses.

Guided Practice

Here is one for you to try on your own.

Use the distributive property to evaluate this expression.

$4(9+2)$

First, we can distribute the four and multiply it by each value in the parentheses. Then we can add.

$36+8=44$

This is our answer.

Practice

Directions: Evaluate each expression using the Distributive Property.

1. 4(3 + 6)
2. 5(2 + 8)
3. 9(12 + 11)
4. 7(8 + 9)
5. 8(7 + 6)
6. 5(12 + 8)
7. 7(9 + 4)
8. 11(2 + 9)
9. 12(12 + 4)
10. 12(9 + 8)
11. 10(9 + 7)
12. 13(2 + 3)
13. 14(8 + 6)
14. 14(9 + 4)
15. 15(5 + 7)

1.5 Square Roots and Irrational Numbers

Here you'll learn how to find and approximate square roots. You'll also learn how to simplify expressions involving square roots.

What if you had a number like 1000 and you wanted to find its square root? After completing this concept, you'll be able to find square roots like this one by hand and with a calculator.

Try This

You can also work out square roots by hand using a method similar to long division. (See the web page at http://www.homeschoolmath.net/teaching/square-root-algorithm.php for an explanation of this method.)

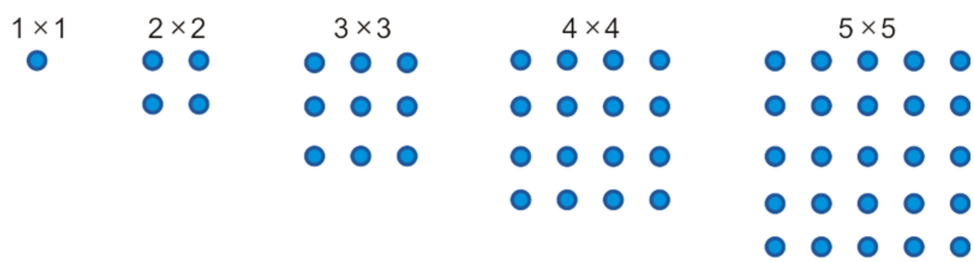
Guidance

The **square root** of a number is a number which, when multiplied by itself, gives the original number. In other words, if $a = b^2$, we say that b is the square root of a.

Note: Negative numbers and positive numbers both yield positive numbers when squared, so each positive number has both a positive and a negative square root. (For example, 3 and -3 can both be squared to yield 9.) The positive square root of a number is called the **principal square root**.

The square root of a number x is written as $\sqrt{x}$ or sometimes as $\sqrt[2]{x}$. The symbol $\sqrt{\ }$ is sometimes called a **radical sign**.

Numbers with whole-number square roots are called **perfect squares**. The first five perfect squares (1, 4, 9, 16, and 25) are shown below.



You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. To find the square root of that number, simply take one of each pair of matching factors and multiply them together.

Example A

Find the principal square root of each of these perfect squares.

a) 121

b) 225

c) 324

Solution

a) $121 = 11 \times 11$, so $\sqrt{121} = 11$.

b) $225 = (5 \times 5) \times (3 \times 3)$, so $\sqrt{225} = 5 \times 3 = 15$.

c) $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$, so $\sqrt{324} = 2 \times 3 \times 3 = 18$.

When the prime factors don't pair up neatly, we "factor out" the ones that do pair up and leave the rest under a radical sign. We write the answer as $a\sqrt{b}$, where a is the product of half the paired factors we pulled out and b is the product of the leftover factors.

Example B

Find the principal square root of the following numbers.

a) 8

b) 48

c) 75

Solution

a) $8 = 2 \times 2 \times 2$. This gives us one pair of 2's and one leftover 2, so $\sqrt{8} = 2\sqrt{2}$.

b) $48 = (2 \times 2) \times (2 \times 2) \times 3$, so $\sqrt{48} = 2 \times 2 \times \sqrt{3}$, or $4\sqrt{3}$.

c) $75 = (5 \times 5) \times 3$, so $\sqrt{75} = 5\sqrt{3}$.

Note that in the last example we collected the paired factors first, **then** we collected the unpaired ones under a single radical symbol. Here are the four rules that govern how we treat square roots.

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$

Example C

Simplify the following square root problems

a) $\sqrt{8} \times \sqrt{2}$

b) $3\sqrt{4} \times 4\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3}$

d) $12\sqrt{10} \div 6\sqrt{5}$

Solution

a) $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$

b) $3\sqrt{4} \times 4\sqrt{3} = 12\sqrt{12} = 12\sqrt{(2 \times 2) \times 3} = 12 \times 2\sqrt{3} = 24\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$

d) $12\sqrt{10} \div 6\sqrt{5} = \frac{12}{6}\sqrt{\frac{10}{5}} = 2\sqrt{2}$

Approximate Square Roots

Terms like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{\ }$ or $\sqrt{x}$ button on a calculator. When the number we plug in is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the answer will be irrational and will look like a random string of digits. Since the calculator can only show some of the infinitely many digits that are actually in the answer, it is really showing us an **approximate answer**—not exactly the right answer, but as close as it can get.

Example D

Use a calculator to find the following square roots. Round your answer to three decimal places.

a) $\sqrt{99}$

b) $\sqrt{5}$

c) $\sqrt{0.5}$

d) $\sqrt{1.75}$

Solution

a) ≈ 9.950

b) ≈ 2.236

c) ≈ 0.707

d) ≈ 1.323

Vocabulary

- The **square root** of a number is a number which gives the original number when multiplied by itself. In algebraic terms, for two numbers a and b, if $a = b^2$, then $b = \sqrt{a}$.
- A square root can have two possible values: a positive value called the **principal square root**, and a negative value (the opposite of the positive value).
- A **perfect square** is a number whose square root is an integer.
- Some mathematical properties of square roots are:
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
  - $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
  - $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
  - $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$
- Square roots of numbers that are not perfect squares (or ratios of perfect squares) are **irrational numbers**. They cannot be written as rational numbers (the ratio of two integers). In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.
- Computing a square root on a calculator will produce an **approximate solution** since the calculator only shows a finite number of digits after the decimal point.

Guided Practice

Find the square root of each number.

a) 576

b) 216

Solution

a) $576 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (3 \times 3)$, so $\sqrt{576} = 2 \times 2 \times 2 \times 3 = 24$.

b) $216 = (2 \times 2) \times 2 \times (3 \times 3) \times 3$, so $\sqrt{216} = 2 \times 3 \times \sqrt{2 \times 3}$, or $6\sqrt{6}$.

Practice

For 1-10, find the following square roots **exactly without using a calculator**, giving your answer in the simplest form.

1. $\sqrt{25}$
2. $\sqrt{24}$
3. $\sqrt{20}$
4. $\sqrt{200}$
5. $\sqrt{2000}$
6. $\sqrt{\frac{1}{4}}$ (Hint: The division rules you learned can be applied backwards!)
7. $\sqrt{\frac{9}{4}}$
8. $\sqrt{0.16}$
9. $\sqrt{0.1}$
10. $\sqrt{0.01}$

For 11-20, use a calculator to find the following square roots. Round to two decimal places.

11. $\sqrt{13}$
12. $\sqrt{99}$
13. $\sqrt{123}$
14. $\sqrt{2}$
15. $\sqrt{2000}$
16. $\sqrt{.25}$
17. $\sqrt{1.35}$
18. $\sqrt{0.37}$
19. $\sqrt{0.7}$
20. $\sqrt{0.01}$

CHAPTER 2

Chapter 2: Equations and Inequalities

Chapter Outline

2.1 Patterns and Equations

Introduction

In mathematics, and especially in algebra, we look for patterns in the numbers we see. The tools of algebra help us describe these patterns with words and with **equations** (formulas or functions). An equation is a mathematical recipe that gives the value of one variable in terms of another.

For example, if a theme park charges $12 admission, then the number of people who enter the park every day and the amount of money taken in by the ticket office are related mathematically, and we can write a rule to find the amount of money taken in by the ticket office.

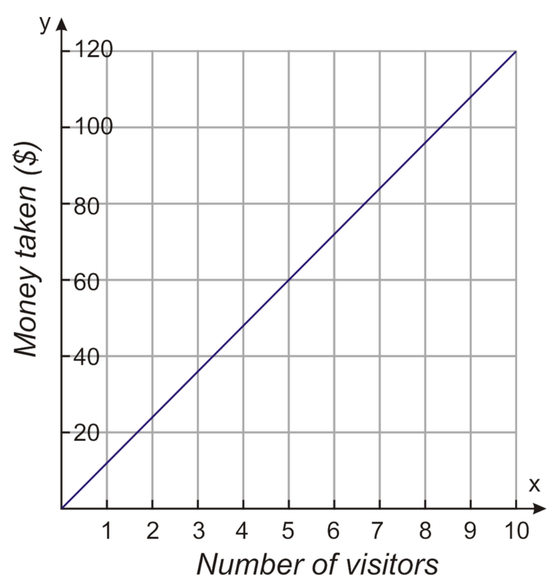
In words, we might say "The amount of money taken in is equal to twelve times the number of people who enter the park."

We could also make a table. The following table relates the number of people who visit the park and the total money taken in by the ticket office.

Number of visitors	1	2	3	4	5	6	7
Money taken in ($)	12	24	36	48	60	72	84

Clearly, we would need a **big** table to cope with a busy day in the middle of a school vacation!

A third way we might relate the two quantities (visitors and money) is with a graph. If we plot the money taken in on the **vertical axis** and the number of visitors on the **horizontal axis**, then we would have a graph that looks like the one shown below. Note that this graph shows a smooth line that includes non-whole number values of x (e.g. $x = 2.5$). In real life this would not make sense, because fractions of people can't visit a park. This is an issue of domain and range, something we will talk about later.



The method we will examine in detail in this lesson is closer to the first way we chose to describe the relationship. In words we said that *"The amount of money taken in is twelve times the number of people who enter the park."* In mathematical terms we can describe this sort of relationship with **variables**. A variable is a letter used to represent an unknown quantity. We can see the beginning of a mathematical formula in the words:

The amount of money taken in **is** twelve **times** the number of people who enter the park.

This can be translated to:

$$\text{the amount of money taken in} = 12 \times (\text{the number of people who enter the park})$$

We can now see which quantities can be assigned to **letters**. First we must state which letters (or **variables**) relate to which quantities. We call this **defining the variables**:

Let $x =$ the number of people who enter the theme park.

Let $y =$ the total amount of money taken in at the ticket office.

We now have a fourth way to describe the relationship: with an algebraic equation.

$$y = 12x$$

Writing a mathematical equation using variables is very convenient. You can perform all of the operations necessary to solve this problem without having to write out the known and unknown quantities over and over again. At the end of the problem, you just need to remember which quantities x and y represent.

Write an Equation

An equation is a term used to describe a collection of **numbers** and **variables** related through mathematical **operators**. An **algebraic equation** will contain letters that represent real quantities. For example, if we wanted to use the algebraic equation in the example above to find the money taken in for a certain number of visitors, we would substitute that number for x and then solve the resulting equation for y.

Example 1

A theme park charges $12 entry to visitors. Find the money taken in if 1296 people visit the park.

Let's break the solution to this problem down into steps. This will be a useful strategy for all the problems in this lesson.

Step 1: Extract the important information.

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors})\\ (\text{number of visitors}) &= 1296\end{aligned}$$

Step 2: Translate into a mathematical equation. To do this, we pick variables to stand for the numbers.

$$\begin{aligned}&\text{Let } y = (\text{number of dollars taken in}).\\ &\text{Let } x = (\text{number of visitors}).\end{aligned}$$

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors})\\ y &= 12 \times x\end{aligned}$$

Step 3: Substitute in any known values for the variables.

$$y = 12 \times x$$
$$x = 1296$$
$$\therefore$$
$$y = 12 \times 1296$$

Step 4: Solve the equation.

$$y = 12 \times 1296 = 15552$$

The amount of money taken in is \$15552.

Step 5: Check the result.

If \$15552 is taken at the ticket office and tickets are \$12, then we can divide the total amount of money collected by the price per individual ticket.

$$(\text{number of people}) = \frac{15552}{12} = 1296$$

1296 is indeed the number of people who entered the park. **The answer checks out.**

Example 2

The following table shows the relationship between two quantities. First, write an equation that describes the relationship. Then, find out the value of b when a is 750.

a	0	10	20	30	40	50
b	20	40	60	80	100	120

Step 1: Extract the important information.

We can see from the table that every time a increases by 10, b increases by 20. However, b is not simply twice the value of a. We can see that when $a = 0,\ b = 20$, and this gives a clue as to what rule the pattern follows. The rule linking a and b is:

"To find b, double the value of a and add 20."

Step 2: Translate into a mathematical equation:

TABLE 2.1:

Text	Translates to	Mathematical Expression
"To find b"	$\rightarrow$	$b =$
"double the value of a"	$\rightarrow$	$2a$
"add 20"	$\rightarrow$	$+\ 20$

Our equation is $b = 2a + 20$.

Step 3: Solve the equation.

The original problem asks for the value of b when a is 750. When a is 750, $b = 2a + 20$ becomes $b = 2(750) + 20$. Following the order of operations, we get:

$$\begin{aligned} b &= 2(750) + 20 \\ &= 1500 + 20 \\ &= 1520 \end{aligned}$$

Step 4: Check the result.

In some cases you can check the result by plugging it back into the original equation. Other times you must simply double-check your math. In either case, checking your answer is *always* a good idea. In this case, we can plug our answer for b into the equation, along with the value for a, and see what comes out. $1520 = 2(750) + 20$ is TRUE because both sides of the equation are equal. A true statement means that **the answer checks out**.

Use a Verbal Model to Write an Equation

In the last example we developed a **rule**, written in words, as a way to develop an algebraic **equation**. We will develop this further in the next few examples.

Example 3

The following table shows the values of two related quantities. Write an equation that describes the relationship mathematically.

TABLE 2.2:

$x-$**value**	$y-$**value**
-2	10
0	0
2	-10
4	-20
6	-30

Step 1: Extract the important information.

We can see from the table that y is five times bigger than x. The value for y is negative when x is positive, and it is positive when x is negative. Here is the rule that links x and y:

"y is the negative of five times the value of x"

Step 2: Translate this statement into a mathematical equation.

TABLE 2.3:

Text	**Translates to**	**Mathematical Expression**
"y is"	$\rightarrow$	$y =$
"negative 5 times the value of x"	$\rightarrow$	$-5x$

Our equation is $y = -5x$.

Step 3: There is nothing in this problem to **solve** for. We can move to Step 4.

Step 4: Check the result.

In this case, the way we would check our answer is to use the equation to generate our own *xy* pairs. If they match the values in the table, then we know our equation is correct. We will plug in -2, 0, 2, 4, and 6 for *x* and solve for *y*:

TABLE 2.4:

x	y
-2	$-5(-2) = 10$
0	$-5(0) = 0$
2	$-5(2) = -10$
4	$-5(4) = -20$
6	$-5(6) = -30$

The $y-$values in this table match the ones in the earlier table. **The answer checks out**.

Example 4

Zarina has a $100 gift card, and she has been spending money on the card in small regular amounts. She checks the balance on the card weekly and records it in the following table.

TABLE 2.5:

Week Number	**Balance ($)**
1	100
2	78
3	56
4	34

Write an equation for the money remaining on the card in any given week.

Step 1: Extract the important information.

The balance remaining on the card is not just a constant multiple of the week number; 100 is 100 times 1, but 78 is not 100 times 2. But there is still a pattern: the balance decreases by 22 whenever the week number increases by 1. This suggests that the balance is somehow related to the amount "-22 times the week number."

In fact, the balance equals "-22 times the week number, plus *something*." To determine what that *something* is, we can look at the values in one row on the table—for example, the first row, where we have a balance of $100 for week number 1.

Step 2: Translate into a mathematical equation.

First, we define our variables. Let *n* stand for the week number and *b* for the balance.

Then we can translate our verbal expression as follows:

TABLE 2.6:

Text	**Translates to**	**Mathematical Expression**
Balance equals -22 times the week number, plus *something*.	$\rightarrow$	$b = -22n + ?$

To find out what that ? represents, we can plug in the values from that first row of the table, where $b = 100$ and $n = 1$. This gives us $100 = -22(1) + ?$.

So what number gives 100 when you add -22 to it? The answer is 122, so that is the number the ? stands for. Now our final equation is:

$$b = -22n + 122$$

Step 3: All we were asked to **find** was the expression. We weren't asked to solve it, so we can move to Step 4.

Step 4: Check the result.

To check that this equation is correct, we see if it really reproduces the data in the table. To do that we plug in values for n:

$$\begin{aligned}
n = 1 &\rightarrow b = -22(1) + 122 = 122 - 22 = 100 \\
n = 2 &\rightarrow b = -22(2) + 122 = 122 - 44 = 78 \\
n = 3 &\rightarrow b = -22(3) + 122 = 122 - 66 = 56 \\
n = 4 &\rightarrow b = -22(4) + 122 = 122 - 88 = 34
\end{aligned}$$

The equation perfectly reproduces the data in the table. **The answer checks out**.

Solve Problems Using Equations

Let's solve the following real-world problem by using the given information to write a mathematical equation that can be solved for a solution.

Example 5

A group of students are in a room. After 25 students leave, it is found that $\frac{2}{3}$ of the original group is left in the room. How many students were in the room at the start?

Step 1: Extract the important information

We know that 25 students leave the room.

We know that $\frac{2}{3}$ of the original number of students are left in the room.

We need to find how many students were in the room at the start.

Step 2: Translate into a mathematical equation. Initially we have an unknown number of students in the room. We can refer to this as the original number.

Let's define the variable $x =$ the original number of students in the room. After 25 students leave the room, the number of students in the room is $x - 25$. We also know that the number of students left is $\frac{2}{3}$ of x. So we have two expressions for the number of students left, and those two expressions are equal because they represent the same number. That means our equation is:

$$\frac{2}{3}x = x - 25$$

Step 3: Solve the equation.

Add 25 to both sides.

$$\begin{aligned}
x - 25 &= \frac{2}{3}x \\
x - 25 + 25 &= \frac{2}{3}x + 25 \\
x &= \frac{2}{3}x + 25
\end{aligned}$$

Subtract $\frac{2}{3}x$ from both sides.

$$x-\frac{2}{3}x=\frac{2}{3}x-\frac{2}{3}x+25$$
$$\frac{1}{3}x=25$$

Multiply both sides by 3.

$$3\cdot\frac{1}{3}x=3\cdot 25$$
$$x=75$$

Remember that x represents the original number of students in the room. So, there were 75 students in the room to start with.

Step 4: Check the answer:

If we start with 75 students in the room and 25 of them leave, then there are $75-25=50$ students left in the room.

$\frac{2}{3}$ of the original number is $\frac{2}{3}\cdot 75=50$.

This means that the number of students who are left over equals $\frac{2}{3}$ of the original number. **The answer checks out**.

The method of defining variables and writing a mathematical equation is the method you will use the most in an algebra course. This method is often used together with other techniques such as making a table of values, creating a graph, drawing a diagram and looking for a pattern.

Review Questions

TABLE 2.7:

Day	Profit
1	20
2	40
3	60
4	80
5	100

1. The above table depicts the profit in dollars taken in by a store each day.
   a. Write a mathematical equation that describes the relationship between the variables in the table.
   b. What is the profit on day 10?
   c. If the profit on a certain day is \$200, what is the profit on the next day?

   (a) Write a mathematical equation that describes the situation: *A full cookie jar has 24 cookies. How many cookies are left in the jar after you have eaten some?*
   (b) How many cookies are in the jar after you have eaten 9 cookies?
   (c) How many cookies are in the jar after you have eaten 9 cookies and then eaten 3 more?

2. Write a mathematical equation for the following situations and solve.
   a. Seven times a number is 35. What is the number?
   b. Three times a number, plus 15, is 24. What is the number?

c. Twice a number is three less than five times another number. Three times the second number is 15. What are the numbers?
d. One number is 25 more than 2 times another number. If each number were multiplied by five, their sum would be 350. What are the numbers?
e. The sum of two consecutive integers is 35. What are the numbers?
f. Peter is three times as old as he was six years ago. How old is Peter?

3. How much water should be added to one liter of pure alcohol to make a mixture of 25% alcohol?
4. A mixture of 50% alcohol and 50% water has 4 liters of water added to it. It is now 25% alcohol. What was the total volume of the original mixture?
5. In Crystal's silverware drawer there are twice as many spoons as forks. If Crystal adds nine forks to the drawer, there will be twice as many forks as spoons. How many forks and how many spoons are in the drawer right now?

(a) Mia drove to Javier's house at 40 miles per hour. Javier's house is 20 miles away. Mia arrived at Javier's house at 2:00 pm. What time did she leave?
(b) Mia left Javier's house at 6:00 pm to drive home. This time she drove 25% faster. What time did she arrive home?
(c) The next day, Mia took the expressway to Javier's house. This route was 24 miles long, but she was able to drive at 60 miles per hour. How long did the trip take?
(d) When Mia took the same route back, traffic on the expressway was 20% slower. How long did the return trip take?

6. The price of an mp3 player decreased by 20% from last year to this year. This year the price of the player is \$120. What was the price last year?
7. SmartCo sells deluxe widgets for \$60 each, which includes the cost of manufacture plus a 20% markup. What does it cost SmartCo to manufacture each widget?
8. Jae just took a math test with 20 questions, each worth an equal number of points. The test is worth 100 points total.

a. Write an equation relating the number of questions Jae got right to the total score he will get on the test.
b. If a score of 70 points earns a grade of $C-$, how many questions would Jae need to get right to get a $C-$ on the test?
c. If a score of 83 points earns a grade of B, how many questions would Jae need to get right to get a B on the test?
d. Suppose Jae got a score of 60% and then was allowed to retake the test. On the retake, he got all the questions right that he got right the first time, and also got half the questions right that he got wrong the first time. What is his new score?

2.2 Variable Expressions

Introduction - The Language of Algebra

No one likes doing the same problem over and over again—that's why mathematicians invented algebra. Algebra takes the basic principles of math and makes them more general, so we can solve a problem once and then use that solution to solve a group of similar problems.

In arithmetic, you've dealt with numbers and their arithmetical operations (such as $+, -, \times, \div$). In algebra, we use symbols called **variables** (which are usually letters, such as $x, y, a, b, c, \ldots$) to represent numbers and sometimes processes.

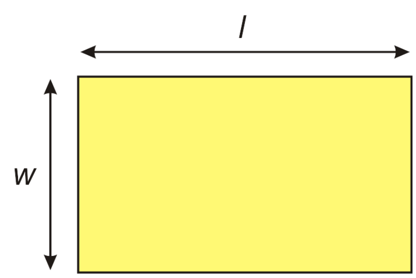
For example, we might use the letter x to represent some number we don't know yet, which we might need to figure out in the course of a problem. Or we might use two letters, like x and y, to show a relationship between two numbers without needing to know what the actual numbers are. The same letters can represent a wide range of possible numbers, and the same letter may represent completely different numbers when used in two different problems.

Using variables offers advantages over solving each problem "from scratch." With variables, we can:

- Formulate arithmetical laws such as $a+b=b+a$ for all real numbers a and b.
- Refer to "unknown" numbers. For instance: find a number x such that $3x+1=10$.
- Write more compactly about functional relationships such as, "If you sell x tickets, then your profit will be $3x-10$ dollars, or "$f(x)=3x-10$," where "f" is the profit function, and x is the input (i.e. how many tickets you sell).

Example 1

Write an algebraic expression for the perimeter and area of the rectangle below.



To find the perimeter, we add the lengths of all 4 sides. We can still do this even if we don't know the side lengths in numbers, because we can use variables like l and w to represent the unknown length and width. If we start at the top left and work clockwise, and if we use the letter P to represent the perimeter, then we can say:

$$P=l+w+l+w$$

We are adding 2 l's and 2 w's, so we can say that:

$$P=2\cdot l+2\cdot w$$

It's customary in algebra to omit multiplication symbols whenever possible. For example, $11x$ means the same thing as $11 \cdot x$ or $11 \times x$. We can therefore also write:

$$P = 2l + 2w$$

Area is *length multiplied by width*. In algebraic terms we get:

$$A = l \times w \to A = l \cdot w \to A = lw$$

Note: $2l + 2w$ by itself is an example of a **variable expression**; $P = 2l + 2w$ is an example of an **equation**. The main difference between expressions and **equa**tions is the presence of an **equa**ls sign (=).

In the above example, we found the simplest possible ways to express the perimeter and area of a rectangle when we don't yet know what its length and width actually are. Now, when we encounter a rectangle whose dimensions we do know, we can simply substitute (or **plug in**) those values in the above equations. In this chapter, we will encounter many expressions that we can evaluate by plugging in values for the variables involved.

Evaluate Algebraic Expressions

When we are given an algebraic expression, one of the most common things we might have to do with it is **evaluate** it for some given value of the variable. The following example illustrates this process.

Example 2

Let $x = 12$*. Find the value of* $2x - 7$*.*

To find the solution, we substitute 12 for x in the given expression. Every time we see x, we replace it with 12.

$$\begin{aligned} 2x - 7 &= 2(12) - 7 \\ &= 24 - 7 \\ &= 17 \end{aligned}$$

Note: At this stage of the problem, we place the substituted value in parentheses. We do this to make the written-out problem easier to follow, and to avoid mistakes. (If we didn't use parentheses and also forgot to add a multiplication sign, we would end up turning $2x$ into 212 instead of 2 times 12!)

Example 3

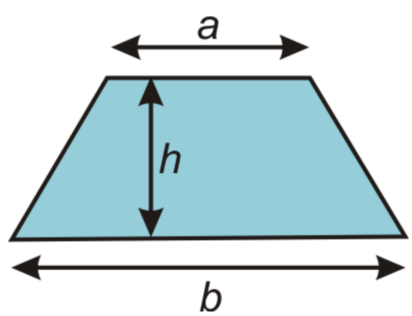
Let $y = -2$*. Find the value of* $\frac{7}{y} - 11y + 2$*.*

Solution

$$\begin{aligned} \frac{7}{(-2)} - 11(-2) + 2 &= -3\frac{1}{2} + 22 + 2 \\ &= 24 - 3\frac{1}{2} \\ &= 20\frac{1}{2} \end{aligned}$$

Many expressions have more than one variable in them. For example, the formula for the perimeter of a rectangle in the introduction has two variables: length (l) and width (w). In these cases, be careful to substitute the appropriate value in the appropriate place.

Example 4



The area of a trapezoid is given by the equation $A = \frac{h}{2}(a+b)$. *Find the area of a trapezoid with bases* $a = 10$ *cm and* $b = 15$ *cm and height* $h = 8$ *cm.*

To find the solution to this problem, we simply take the values given for the variables $a,\ b$, and h, and plug them in to the expression for A:

$$\begin{aligned}
A &= \frac{h}{2}(a+b) && \text{Substitute 10 for } a,\ 15 \text{ for } b,\ \text{and 8 for } h.\\
A &= \frac{8}{2}(10+15) && \text{Evaluate piece by piece. } 10+15=25;\ \frac{8}{2}=4.\\
A &= 4(25) = 100
\end{aligned}$$

Solution: *The area of the trapezoid is 100 square centimeters.*

Evaluate Algebraic Expressions with Exponents

Many formulas and equations in mathematics contain exponents. Exponents are used as a short-hand notation for repeated multiplication. For example:

$$\begin{aligned}
2 \cdot 2 &= 2^2\\
2 \cdot 2 \cdot 2 &= 2^3
\end{aligned}$$

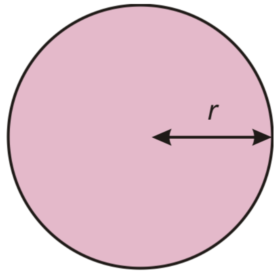
The exponent stands for how many times the number is used as a factor (multiplied). When we deal with integers, it is usually easiest to simplify the expression. We simplify:

$$\begin{aligned}
2^2 &= 4\\
2^3 &= 8
\end{aligned}$$

However, we need exponents when we work with variables, because it is much easier to write x^8 than $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.

To evaluate expressions with exponents, substitute the values you are given for each variable and simplify. It is especially important in this case to substitute using parentheses in order to make sure that the simplification is done correctly.

Example 5



The area of a circle is given by the formula $A = \pi r^2$. Find the area of a circle with radius $r = 17$ inches.

Substitute values into the equation.

$$\begin{aligned} A &= \pi r^2 && \text{Substitute 17 for } r. \\ A &= \pi(17)^2 && \pi \cdot 17 \cdot 17 \approx 907.9202\ldots \text{ Round to 2 decimal places.} \end{aligned}$$

The area is approximately 907.92 square inches.

Example 6

Find the value of $\frac{x^2y^3}{x^3+y^2}$, for $x = 2$ and $y = -4$.

Substitute the values of x and y in the following.

$$\frac{x^2y^3}{x^3+y^2} = \frac{(2)^2(-4)^3}{(2)^3+(-4)^2} \qquad \text{Substitute 2 for } x \text{ and } -4 \text{ for } y.$$

$$\frac{4(-64)}{8+16} = \frac{-256}{24} = \frac{-32}{3} \qquad \text{Evaluate expressions: } (2)^2 = (2)(2) = 4 \text{ and}$$

$(2)^3 = (2)(2)(2) = 8$. $(-4)^2 = (-4)(-4) = 16$ and

$(-4)^3 = (-4)(-4)(-4) = -64$.

Example 7

The height (h) of a ball in flight is given by the formula $h = -32t^2 + 60t + 20$, where the height is given in feet and the time (t) is given in seconds. Find the height of the ball at time $t = 2$ seconds.

Solution

$$\begin{aligned} h &= -32t^2 + 60t + 20 \\ &= -32(2)^2 + 60(2) + 20 && \text{Substitute 2 for } t. \\ &= -32(4) + 60(2) + 20 \\ &= 12 \end{aligned}$$

The height of the ball is 12 feet.

Review Questions

1. Write the following in a more condensed form by leaving out a multiplication symbol.
   a. $2 \times 11x$

b. $1.35 \cdot y$
c. $3 \times \frac{1}{4}$
d. $\frac{1}{4} \cdot z$

2. Evaluate the following expressions for $a = -3,\ b = 2,\ c = 5$, and $d = -4$.
   a. $2a + 3b$
   b. $4c + d$
   c. $5ac - 2b$
   d. $\frac{2a}{c-d}$
   e. $\frac{3b}{d}$
   f. $\frac{a-4b}{3c+2d}$
   g. $\frac{1}{a+b}$
   h. $\frac{ab}{cd}$
3. Evaluate the following expressions for $x = -1,\ y = 2,\ z = -3$, and $w = 4$.
   a. $8x^3$
   b. $\frac{5x^2}{6z^3}$
   c. $3z^2 - 5w^2$
   d. $x^2 - y^2$
   e. $\frac{z^3+w^3}{z^3-w^3}$
   f. $2x^3 - 3x^2 + 5x - 4$
   g. $4w^3 + 3w^2 - w + 2$
   h. $3 + \frac{1}{z^2}$
4. The weekly cost C of manufacturing x remote controls is given by the formula $C = 2000 + 3x$, where the cost is given in dollars.
   a. What is the cost of producing 1000 remote controls?
   b. What is the cost of producing 2000 remote controls?
   c. What is the cost of producing 2500 remote controls?
5. The volume of a box without a lid is given by the formula $V = 4x(10-x)^2$, where x is a length in inches and V is the volume in cubic inches.
   a. What is the volume when $x = 2$?
   b. What is the volume when $x = 3$?

2.3 Equations and Inequalities

Introduction

In algebra, an **equation** is a mathematical expression that contains an equals sign. It tells us that two expressions represent the same number. For example, $y = 12x$ is an equation. An **inequality** is a mathematical expression that contains inequality signs. For example, $y \leq 12x$ is an inequality. Inequalities are used to tell us that an expression is either larger or smaller than another expression. Equations and inequalities can contain both **variables** and **constants**.

Variables are usually given a letter and they are used to represent unknown values. These quantities can change because they depend on other numbers in the problem.

Constants are quantities that remain unchanged. Ordinary numbers like 2, -3, $\frac{3}{4}$, and π are constants.

Equations and inequalities are used as a shorthand notation for situations that involve numerical data. They are very useful because most problems require several steps to arrive at a solution, and it becomes tedious to repeatedly write out the situation in words.

Write Equations and Inequalities

Here are some examples of equations:

$$3x-2=5 \qquad x+9=2x+5 \qquad \frac{x}{3}=15 \qquad x^2+1=10$$

To write an inequality, we use the following symbols:

$>$**greater than**

$\geq$ **greater than or equal to**

$<$**less than**

$\leq$ **less than or equal to**

$\neq$ **not equal to**

Here are some examples of inequalities:

$$3x<5 \qquad 4-x \leq 2x \qquad x^2+2x-1>0 \qquad \frac{3x}{4} \geq \frac{x}{2}-3$$

The most important skill in algebra is the ability to translate a word problem into the correct equation or inequality so you can find the solution easily. The first two steps are **defining the variables** and **translating** the word problem into a mathematical equation.

Defining the variables means that we assign letters to any unknown quantities in the problem.

Translating means that we change the word expression into a mathematical expression containing variables and mathematical operations with an equal sign or an inequality sign.

Example 1

Define the variables and translate the following expressions into equations.

a) A number plus 12 is 20.

b) 9 less than twice a number is 33.

c) $20 was one quarter of the money spent on the pizza.

Solution

a) **Define**

Let $n =$ the number we are seeking.

Translate

A number plus 12 is 20.

$$n+12=20$$

b) **Define**

Let $n =$ the number we are seeking.

Translate

9 less than twice a number is 33.

This means that twice the number, minus 9, is 33.

$$2n-9=33$$

c) **Define**

Let $m =$ the money spent on the pizza.

Translate

$20 was one quarter of the money spent on the pizza.

$$20=\frac{1}{4}m$$

Often word problems need to be reworded before you can write an equation.

Example 2

Find the solution to the following problems.

a) Shyam worked for two hours and packed 24 boxes. How much time did he spend on packing one box?

b) After a 20% discount, a book costs $12. How much was the book before the discount?

Solution

a) **Define**

Let $t =$ time it takes to pack one box.

Translate

Shyam worked for two hours and packed 24 boxes. This means that two hours is 24 times the time it takes to pack one box.

$$2 = 24t$$

Solve

$$t = \frac{2}{24} = \frac{1}{12} \text{ hours}$$
$$\frac{1}{12} \times 60 \text{ minutes} = 5 \text{ minutes}$$

Answer

Shyam takes 5 minutes to pack a box.

b) **Define**

Let $p =$ the price of the book before the discount.

Translate

After a 20% discount, the book costs \$12. This means that the price minus 20% of the price is \$12.

$$p - 0.20p = 12$$

Solve

$$p - 0.20p = 0.8p, \text{ so } 0.8p = 12$$
$$p = \frac{12}{0.8} = 15$$

Answer

The price of the book before the discount was \$15.

Check

If the original price was \$15, then the book was discounted by 20% of \$15, or \$3. $\$15 - 3 = \$12$. **The answer checks out**.

Example 3

Define the variables and translate the following expressions into inequalities.

a) The sum of 5 and a number is less than or equal to 2.

b) The distance from San Diego to Los Angeles is less than 150 miles.

c) Diego needs to earn more than an 82 on his test to receive a B in his algebra class.

d) A child needs to be 42 inches or more to go on the roller coaster.

Solution

a) **Define**

Let $n =$ the unknown number.

Translate

$$5+n \leq 2$$

b) **Define**

Let $d =$ the distance from San Diego to Los Angeles in miles.

Translate

$$d < 150$$

c) **Define**

Let $x =$ Diego's test grade.

Translate

$$x > 82$$

d) **Define**

Let $h =$ the height of child in inches.

Translate:

$$h \geq 42$$

Check Solutions to Equations

You will often need to check solutions to equations in order to check your work. In a math class, checking that you arrived at the correct solution is very good practice. We check the solution to an equation by replacing the variable in an equation with the value of the solution. A solution should result in a true statement when plugged into the equation.

Example 4

Check that the given number is a solution to the corresponding equation.

a) $y = -1;\ 3y+5 = -2y$

b) $z = 3;\ z^2+2z = 8$

c) $x = -\frac{1}{2};\ 3x+1 = x$

Solution

Replace the variable in each equation with the given value.

a)

$$\begin{aligned} 3(-1)+5 &= -2(-1) \\ -3+5 &= 2 \\ 2 &= 2 \end{aligned}$$

This is a true statement. This means that $y = -1$ is a solution to $3y+5 = -2y$.

b)

$$\begin{aligned} 3^2 + 2(3) &= 8 \\ 9+6 &= 8 \\ 15 &= 8 \end{aligned}$$

This is not a true statement. This means that $z = 3$ is **not a solution** to $z^2 + 2z = 8$.

c)

$$\begin{aligned} 3\left(-\frac{1}{2}\right) + 1 &= -\frac{1}{2} \\ \left(-\frac{3}{2}\right) + 1 &= -\frac{1}{2} \\ -\frac{1}{2} &= -\frac{1}{2} \end{aligned}$$

This is a true statement. This means that $x = -\frac{1}{2}$ is a solution to $3x+1 = x$.

Check Solutions to Inequalities

To check the solution to an inequality, we replace the variable in the inequality with the value of the solution. A solution to an inequality produces a true statement when substituted into the inequality.

Example 5

Check that the given number is a solution to the corresponding inequality.

a) $a = 10$; $20a \leq 250$

b) $b = -0.5$; $\frac{3-b}{b} > -4$

c) $x = \frac{3}{4}$; $4x+5 \leq 8$

Solution

Replace the variable in each inequality with the given value.

a)

$$\begin{aligned} 20(10) &\leq 250 \\ 200 &\leq 250 \end{aligned}$$

This statement is true. This means that $a = 10$ is a solution to the inequality $20a \leq 250$.

Note that $a = 10$ is not the only solution to this inequality. If we divide both sides of the inequality by 20, we can write it as $a \leq 12.5$. This means that any number less than or equal to 12.5 is also a solution to the inequality.

b)

$$\begin{aligned} \frac{3-(-0.5)}{(-0.5)} &> -4 \\ \frac{3+0.5}{-0.5} &> -4 \\ -\frac{3.5}{0.5} &> -4 \\ -7 &> -4 \end{aligned}$$

This statement is false. This means that $b = -0.5$ is not a solution to the inequality $\frac{3-b}{b} > -4$.

c)

$$\begin{aligned} 4\left(\frac{3}{4}\right)+5 &\geq 8 \\ 3+5 &\geq 8 \\ 8 &\geq 8 \end{aligned}$$

This statement is true. It is true because this inequality includes an equals sign; since 8 is equal to itself, it is also "greater than or equal to" itself. This means that $x = \frac{3}{4}$ is a solution to the inequality $4x+5 \leq 8$.

Solve Real-World Problems Using an Equation

Let's use what we have learned about defining variables, writing equations and writing inequalities to solve some real-world problems.

Example 6

Tomatoes cost $0.50 each and avocados cost $2.00 each. Anne buys six more tomatoes than avocados. Her total bill is $8. How many tomatoes and how many avocados did Anne buy?

Solution

Define

Let $a =$ the number of avocados Anne buys.

Translate

Anne buys six more tomatoes than avocados. This means that $a+6 =$ the number of tomatoes.

Tomatoes cost $0.50 each and avocados cost $2.00 each. Her total bill is $8. This means that .50 times the number of tomatoes plus 2 times the number of avocados equals 8.

$$\begin{aligned} 0.5(a+6)+2a &= 8 \\ 0.5a+0.5\cdot 6+2a &= 8 \\ 2.5a+3 &= 8 \\ 2.5a &= 5 \\ a &= 2 \end{aligned}$$

Remember that $a =$ the number of avocados, so Anne buys two avocados. The number of tomatoes is $a+6 = 2+6 = 8$.

Answer

Anne bought 2 avocados and 8 tomatoes.

Check

If Anne bought two avocados and eight tomatoes, the total cost is: $(2\times 2)+(8\times 0.5) = 4+4 = 8$. **The answer checks out.**

Example 7

To organize a picnic Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs. What is the possible number of hamburgers Peter has?

Solution

Define

Let $x =$ number of hamburgers

Translate

Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs.

This means that twice the number of hot dogs is less than or equal to the number of hamburgers.

$$2 \times 24 \leq x, \text{ or } 48 \leq x$$

Answer

Peter needs at least 48 hamburgers.

Check

48 hamburgers is twice the number of hot dogs. So more than 48 hamburgers is more than twice the number of hot dogs. **The answer checks out**.

Review Questions

1. Define the variables and translate the following expressions into equations.
   a. Peter's Lawn Mowing Service charges \$10 per job and \$0.20 per square yard. Peter earns \$25 for a job.
   b. Renting the ice-skating rink for a birthday party costs \$200 plus \$4 per person. The rental costs \$324 in total.
   c. Renting a car costs \$55 per day plus \$0.45 per mile. The cost of the rental is \$100.
   d. Nadia gave Peter 4 more blocks than he already had. He already had 7 blocks.
2. Define the variables and translate the following expressions into inequalities.
   a. A bus can seat 65 passengers or fewer.
   b. The sum of two consecutive integers is less than 54.
   c. The product of a number and 3 is greater than 30.
   d. An amount of money is invested at 5% annual interest. The interest earned at the end of the year is greater than or equal to \$250.
   e. You buy hamburgers at a fast food restaurant. A hamburger costs \$0.49. You have at most \$3 to spend. Write an inequality for the number of hamburgers you can buy.
   f. Mariel needs at least 7 extra credit points to improve her grade in English class. Additional book reports are worth 2 extra credit points each. Write an inequality for the number of book reports Mariel needs to do.
3. Check whether the given number is a solution to the corresponding equation.
   a. $a = -3;\ 4a + 3 = -9$
   b. $x = \frac{4}{3};\ \frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$
   c. $y = 2;\ 2.5y - 10.0 = -5.0$
   d. $z = -5;\ 2(5 - 2z) = 20 - 2(z - 1)$
4. Check whether the given number is a solution to the corresponding inequality.
   a. $x = 12;\ 2(x + 6) \leq 8x$
   b. $z = -9;\ 1.4z + 5.2 > 0.4z$

c. $y = 40;\ -\frac{5}{2}y + \frac{1}{2} < -18$
d. $t = 0.4;\ 80 \geq 10(3t + 2)$

5. The cost of a Ford Focus is 27% of the price of a Lexus GS 450h. If the price of the Ford is \$15000, what is the price of the Lexus?
6. On your new job you can be paid in one of two ways. You can either be paid \$1000 per month plus 6% commission of total sales or be paid \$1200 per month plus 5% commission on sales over \$2000. For what amount of sales is the first option better than the second option? Assume there are always sales over \$2000.
7. A phone company offers a choice of three text-messaging plans. Plan A gives you unlimited text messages for \$10 a month; Plan B gives you 60 text messages for \$5 a month and then charges you \$0.05 for each additional message; and Plan C has no monthly fee but charges you \$0.10 per message.

   a. If m is the number of messages you send per month, write an expression for the monthly cost of each of the three plans.
   b. For what values of m is Plan A cheaper than Plan B?
   c. For what values of m is Plan A cheaper than Plan C?
   d. For what values of m is Plan B cheaper than Plan C?
   e. For what values of m is Plan A the cheapest of all? (Hint: for what values is A both cheaper than B and cheaper than C?)
   f. For what values of m is Plan B the cheapest of all? (Careful—for what values is B cheaper than A?)
   g. For what values of m is Plan C the cheapest of all?
   h. If you send 30 messages per month, which plan is cheapest?
   i. What is the cost of each of the three plans if you send 30 messages per month?

2.4 1-Step Equations

Introduction

Nadia is buying a new mp3 player. Peter watches her pay for the player with a \$100 bill. She receives \$22.00 in change, and from only this information, Peter works out how much the player cost. How much was the player?

In algebra, we can solve problems like this using an **equation**. An **equation** is an algebraic expression that involves an **equals** sign. If we use the letter x to represent the cost of the mp3 player, we can write the equation $x+22=100$. This tells us that the value of the player **plus** the value of the change received is **equal** to the \$100 that Nadia paid.

Another way we could write the equation would be $x=100-22$. This tells us that the value of the player is **equal** to the total amount of money Nadia paid $(100-22)$. This equation is mathematically equivalent to the first one, but it is easier to solve.

In this chapter, we will learn how to solve for the variable in a one-variable linear equation. **Linear equations** are equations in which each term is either a constant, or a constant times a single variable (raised to the first power). The term linear comes from the word line, because the graph of a linear equation is always a line.

We'll start with simple problems like the one in the last example.

Solving Equations Using Addition and Subtraction

When we work with an algebraic equation, it's important to remember that the two sides have to stay equal for the equation to stay true. We can change the equation around however we want, but whatever we do to one side of the equation, we have to do to the other side. In the introduction above, for example, we could get from the first equation to the second equation by subtracting 22 from both sides:

$$\begin{aligned} x+22 &= 100 \\ x+22-22 &= 100-22 \\ x &= 100-22 \end{aligned}$$

Similarly, we can add numbers to each side of an equation to help solve for our unknown.

Example 1

Solve $x-3=9$.

Solution

To solve an equation for x, we need to **isolate** x—that is, we need to get it by itself on one side of the equals sign. Right now our x has a 3 subtracted from it. To reverse this, we'll add 3—but we must add 3 to **both sides.**

$$\begin{aligned} x-3 &= 9 \\ x-3+3 &= 9+3 \\ x+0 &= 9+3 \\ x &= 12 \end{aligned}$$

Example 2

Solve $z - 9.7 = -1.026$

Solution

It doesn't matter what the variable is—the solving process is the same.

$$\begin{aligned} z - 9.7 &= -1.026 \\ z - 9.7 + 9.7 &= -1.026 + 9.7 \\ z &= 8.674 \end{aligned}$$

Make sure you understand the addition of decimals in this example!

Example 3

Solve $x + \frac{4}{7} = \frac{9}{5}$.

Solution

To isolate x, we need to subtract $\frac{4}{7}$ from both sides.

$$\begin{aligned} x + \frac{4}{7} &= \frac{9}{5} \\ x + \frac{4}{7} - \frac{4}{7} &= \frac{9}{5} - \frac{4}{7} \\ x &= \frac{9}{5} - \frac{4}{7} \end{aligned}$$

Now we have to subtract fractions, which means we need to find the LCD. Since 5 and 7 are both prime, their lowest common multiple is just their product, 35.

$$\begin{aligned} x &= \frac{9}{5} - \frac{4}{7} \\ x &= \frac{7 \cdot 9}{7 \cdot 5} - \frac{4 \cdot 5}{7 \cdot 5} \\ x &= \frac{63}{35} - \frac{20}{35} \\ x &= \frac{63 - 20}{35} \\ x &= \frac{43}{35} \end{aligned}$$

Make sure you're comfortable with decimals and fractions! To master algebra, you'll need to work with them frequently.

Solving Equations Using Multiplication and Division

Suppose you are selling pizza for \$1.50 a slice and you can get eight slices out of a single pizza. How much money do you get for a single pizza? It shouldn't take you long to figure out that you get $8 \times \$1.50 = \$12.00$. You solved this problem by multiplying. Here's how to do the same thing algebraically, using x to stand for the cost in dollars of the whole pizza.

Example 4

Solve $\frac{1}{8} \cdot x = 1.5$.

Our x is being multiplied by one-eighth. To cancel that out and get x by itself, we have to multiply by the reciprocal, 8. Don't forget to multiply **both sides** of the equation.

$$8\left(\frac{1}{8} \cdot x\right) = 8(1.5)$$
$$x = 12$$

Example 5

Solve $\frac{9x}{5} = 5$.

$\frac{9x}{5}$ is equivalent to $\frac{9}{5} \cdot x$, so to cancel out that $\frac{9}{5}$, we multiply by the reciprocal, $\frac{5}{9}$.

$$\frac{5}{9}\left(\frac{9x}{5}\right) = \frac{5}{9}(5)$$
$$x = \frac{25}{9}$$

Example 6

Solve $0.25x = 5.25$.

0.25 is the decimal equivalent of one fourth, so to cancel out the 0.25 factor we would multiply by 4.

$$4(0.25x) = 4(5.25)$$
$$x = 21$$

Solving by division is another way to isolate x. Suppose you buy five identical candy bars, and you are charged \$3.25. How much did each candy bar cost? You might just divide \$3.25 by 5, but let's see how this problem looks in algebra.

Example 7

Solve $5x = 3.25$.

To cancel the 5, we divide both sides by 5.

$$\frac{5x}{5} = \frac{3.25}{5}$$
$$x = 0.65$$

Example 8

Solve $7x = \frac{5}{11}$.

Divide both sides by 7.

$$x = \frac{5}{11.7}$$
$$x = \frac{5}{77}$$

Example 9

Solve $1.375x = 1.2$.

Divide by 1.375

$$x = \frac{1.2}{1.375}$$
$$x = 0.8\overline{72}$$

Notice the bar above the final two decimals; it means that those digits recur, or repeat. The full answer is 0.872727272727272....

Solve Real-World Problems Using Equations

Example 10

In the year 2017, Anne will be 45years old. In what year was Anne born?

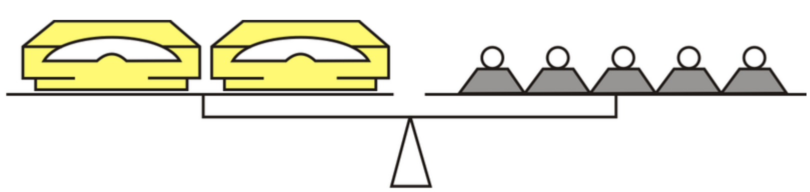
The unknown here is the year Anne was born, so that's our variable x. Here's our equation:

$$x + 45 = 2017$$
$$x + 45 - 45 = 2017 - 45$$
$$x = 1972$$

Anne was born in 1972.

Example 11

A mail order electronics company stocks a new mini DVD player and is using a balance to determine the shipping weight. Using only one-pound weights, the shipping department found that the following arrangement balances:



How much does each DVD player weigh?

Solution

Since the system balances, the total weight on each side must be equal. To write our equation, we'll use x for the weight of one DVD player, which is unknown. There are two DVD players, weighing a total of $2x$ pounds, on the left side of the balance, and on the right side are 5 1-pound weights, weighing a total of 5 pounds. So our equation is $2x = 5$. Dividing both sides by 2 gives us $x = 2.5$.

Each DVD player weighs 2.5 pounds.

Example 12

In 2004, Takeru Kobayashi of Nagano, Japan, ate 53.5 hot dogs in 12 minutes. This was 3 more hot dogs than his own previous world record, set in 2002. Calculate:

a) *How many minutes it took him to eat one hot dog.*

b) *How many hot dogs he ate per minute.*

c) *What his old record was.*

Solution

a) We know that the total time for 53.5 hot dogs is 12 minutes. We want to know the time for one hot dog, so that's x. Our equation is $53.5x = 12$. Then we divide both sides by 53.5 to get $x = \frac{12}{53.5}$, or $x = 0.224\ minutes$.

We can also multiply by 60 to get the time in seconds; 0.224 minutes is about 13.5 seconds. So that's how long it took Takeru to eat one hot dog.

b) Now we're looking for hot dogs per minute instead of minutes per hot dog. We'll use the variable y instead of x this time so we don't get the two confused. 12 minutes, times the number of hot dogs per minute, equals the total number of hot dogs, so $12y = 53.5$. Dividing both sides by 12 gives us $y = \frac{53.5}{12}$, or $y = 4.458$ hot dogs per minute.

c) We know that his new record is 53.5, and we know that's three more than his old record. If we call his old record z, we can write the following equation: $z + 3 = 53.5$. Subtracting 3 from both sides gives us $z = 50.5$. So Takeru's old record was 50.5 hot dogs in 12 minutes.

Lesson Summary

- An equation in which each term is either a constant or the product of a constant and a single variable is a **linear equation**.
- We can add, subtract, multiply, or divide both sides of an equation by the same value and still have an equivalent equation.
- To solve an equation, **isolate** the unknown variable on one side of the equation by applying one or more arithmetic operations to both sides.

Review Questions

1. Solve the following equations for x.

    a. $x = 11 = 7$
    b. $x - 1.1 = 3.2$
    c. $7x = 21$
    d. $4x = 1$
    e. $\frac{5x}{12} = \frac{2}{3}$
    f. $x + \frac{5}{2} = \frac{2}{3}$
    g. $x - \frac{5}{6} = \frac{3}{8}$

h. $0.01x = 11$

2. Solve the following equations for the unknown variable.

a. $q - 13 = -13$
b. $z + 1.1 = 3.0001$
c. $21s = 3$
d. $t + \frac{1}{2} = \frac{1}{3}$
e. $\frac{7f}{11} = \frac{7}{11}$
f. $\frac{3}{4} = -\frac{1}{2} - y$
g. $6r = \frac{3}{8}$
h. $\frac{9b}{16} = \frac{3}{8}$

3. Peter is collecting tokens on breakfast cereal packets in order to get a model boat. In eight weeks he has collected 10 tokens. He needs 25 tokens for the boat. Write an equation and determine the following information.

a. How many more tokens he needs to collect, n.
b. How many tokens he collects per week, w.
c. How many more weeks remain until he can send off for his boat, r.

4. Juan has baked a cake and wants to sell it in his bakery. He is going to cut it into 12 slices and sell them individually. He wants to sell it for three times the cost of making it. The ingredients cost him \$8.50, and he allowed \$1.25 to cover the cost of electricity to bake it. Write equations that describe the following statements

a. The amount of money that he sells the cake for (u).
b. The amount of money he charges for each slice (c).
c. The total profit he makes on the cake (w).

5. Jane is baking cookies for a large party. She has a recipe that will make one batch of two dozen cookies, and she decides to make five batches. To make five batches, she finds that she will need 12.5 cups of flour and 15 eggs.

a. How many cookies will she make in all?
b. How many cups of flour go into one batch?
c. How many eggs go into one batch?
d. If Jane only has a dozen eggs on hand, how many more does she need to make five batches?
e. If she doesn't go out to get more eggs, how many batches can she make? How many cookies will that be?

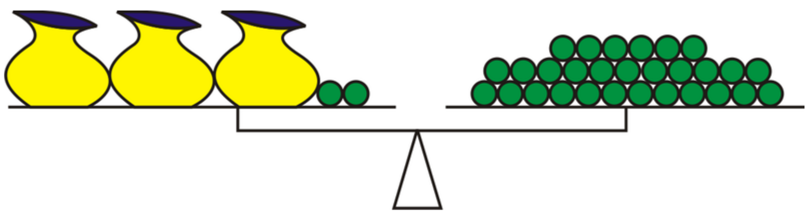
2.5 2-Step Equations

Solve a Two-Step Equation

We've seen how to solve for an unknown by isolating it on one side of an equation and then evaluating the other side. Now we'll see how to solve equations where the variable takes more than one step to isolate.

Example 1

Rebecca has three bags containing the same number of marbles, plus two marbles left over. She places them on one side of a balance. Chris, who has more marbles than Rebecca, adds marbles to the other side of the balance. He finds that with 29 marbles, the scales balance. How many marbles are in each bag? Assume the bags weigh nothing.



Solution

We know that the system balances, so the weights on each side must be equal. If we use x to represent the number of marbles in each bag, then we can see that on the left side of the scale we have three bags (each containing x marbles) plus two extra marbles, and on the right side of the scale we have 29 marbles. The balancing of the scales is similar to the balancing of the following equation.

$$3x+2=29$$

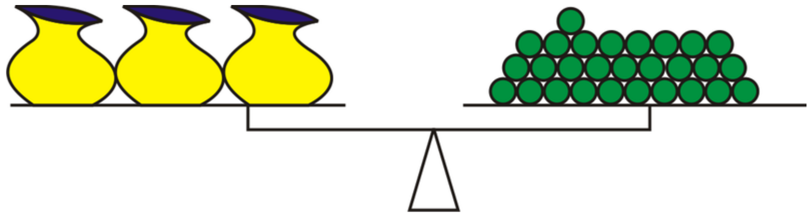
"Three bags plus two marbles **equals** 29 marbles"

To solve for x, we need to first get all the variables (terms containing an x) alone on one side of the equation. We've already got all the x's on one side; now we just need to isolate them.

$$\begin{aligned}
3x+2&=29 \\
3x+2-2&=29-2 && \text{Get rid of the 2 on the left by subtracting it from both sides.} \\
3x&=27 \\
\frac{3x}{3}&=\frac{27}{3} && \text{Divide both sides by 3.} \\
x&=9
\end{aligned}$$

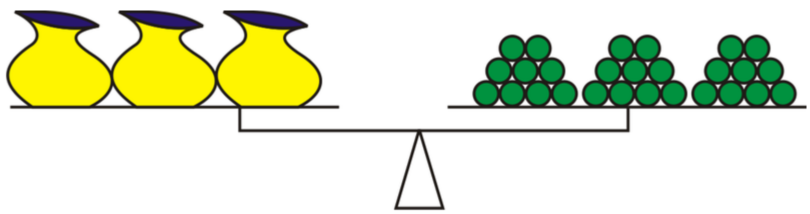
There are nine marbles in each bag.

We can do the same with the real objects as we did with the equation. Just as we subtracted 2 from both sides of the equals sign, we could remove two marbles from each side of the scale. Because we removed the same number of marbles from each side, we know the scales will still balance.



Then, because there are three bags of marbles on the left-hand side of the scale, we can divide the marbles on the right-hand side into three equal piles. You can see that there are nine marbles in each.

Three bags of marbles ***balances*** *three piles of nine marbles.*



So each bag of marbles balances nine marbles, meaning that each bag contains nine marbles.

Example 2

Solve $6(x+4) = 12$.

This equation has the x buried in parentheses. To dig it out, we can proceed in one of two ways: we can either distribute the six on the left, or divide both sides by six to remove it from the left. Since the right-hand side of the equation is a multiple of six, it makes sense to divide. That gives us $x+4 = 2$. Then we can subtract 4 from both sides to get $x = -2$.

Example 3

Solve $\frac{x-3}{5} = 7$.

It's always a good idea to get rid of fractions first. Multiplying both sides by 5 gives us $x-3 = 35$, and then we can add 3 to both sides to get $x = 38$.

Example 4

Solve $\frac{5}{9}(x+1) = \frac{2}{7}$.

First, we'll cancel the fraction on the left by multiplying by the reciprocal (the multiplicative inverse).

$$\begin{aligned}\frac{9}{5} \cdot \frac{5}{9}(x+1) &= \frac{9}{5} \cdot \frac{2}{7}\\(x+1) &= \frac{18}{35}\end{aligned}$$

Then we subtract 1 from both sides. ($\frac{35}{35}$ is equivalent to 1.)

$$\begin{aligned}x+1&=\frac{18}{35}\\x+1-1&=\frac{18}{35}-\frac{35}{35}\\x&=\frac{18-35}{35}\\x&=\frac{-17}{35}\end{aligned}$$

These examples are called **two-step equations**, because we need to perform two separate operations on the equation to isolate the variable.

Solve a Two-Step Equation by Combining Like Terms

When we look at a linear equation we see two kinds of terms: those that contain the unknown variable, and those that don't. When we look at an equation that has an x on both sides, we know that in order to solve it, we need to get all the $x-$terms on one side of the equation. This is called **combining like terms**. The terms with an x in them are **like terms** because they contain the same variable (or, as you will see in later chapters, the same combination of variables).

TABLE 2.8:

Like Terms	Unlike Terms
$4x, 10x, -3.5x$, and $\frac{x}{12}$	$3x$ and $3y$
$3y, 0.000001y$, and y	$4xy$ and $4x$
$xy, 6xy$, and $2.39xy$	$0.5x$ and 0.5

To add or subtract like terms, we can use the Distributive Property of Multiplication.

$$\begin{aligned}3x+4x&=(3+4)x=7x\\0.03xy-0.01xy&=(0.03-0.01)xy=0.02xy\\-y+16y+5y&=(-1+16+5)y=10y\\5z+2z-7z&=(5+2-7)z=0z=0\end{aligned}$$

To solve an equation with two or more like terms, we need to combine the terms first.

Example 5

Solve $(x+5)-(2x-3)=6$.

There are two like terms: the x and the $-2x$ (don't forget that the negative sign applies to everything in the parentheses). So we need to get those terms together. The associative and distributive properties let us rewrite the equation as $x+5-2x+3=6$, and then the commutative property lets us switch around the terms to get $x-2x+5+3=6$, or $(x-2x)+(5+3)=6$.

$(x-2x)$ is the same as $(1-2)x$, or $-x$, so our equation becomes $-x+8=6$

Subtracting 8 from both sides gives us $-x=-2$.

And finally, multiplying both sides by -1 gives us $x=2$.

Example 6

Solve $\frac{x}{2} - \frac{x}{3} = 6$.

This problem requires us to deal with fractions. We need to write all the terms on the left over a common denominator of six.

$$\frac{3x}{6} - \frac{2x}{6} = 6$$

Then we subtract the fractions to get $\frac{x}{6} = 6$.

Finally we multiply both sides by 6 to get $x = 36$.

Solve Real-World Problems Using Two-Step Equations

The hardest part of solving word problems is translating from words to an equation. First, you need to look to see what the equation is asking. What is the **unknown** for which you have to solve? That will be what your **variable** stands for. Then, follow what is going on with your variable all the way through the problem.

Example 7

An emergency plumber charges \$65 as a call-out fee plus an additional \$75 per hour. He arrives at a house at 9:30 and works to repair a water tank. If the total repair bill is \$196.25, at what time was the repair completed?

In order to solve this problem, we collect the information from the text and convert it to an equation.

Unknown: time taken in hours –this will be our x

The bill is made up of two parts: a call out fee and a per-hour fee. The call out is a flat fee, and independent of x—it's the same no matter how many hours the plumber works. The per-hour part depends on the number of hours (x). So the total fee is \$65 (no matter what) plus \$75x (where x is the number of hours), or $65 + 75x$.

Looking at the problem again, we also can see that the total bill is \$196.25. So our final equation is $196.25 = 65 + 75x$.

Solving for x:

$$\begin{aligned}
196.25 &= 65 + 75x && \text{Subtract 65 from both sides.}\\
131.25 &= 75x && \text{Divide both sides by 75.}\\
1.75 &= x && \text{The job took 1.75 hours.}
\end{aligned}$$

Solution

The repair job was completed 1.75 hours after 9:30, so it was completed at 11:15AM.

Example 8

When Asia was young her Daddy marked her height on the door frame every month. Asia's Daddy noticed that between the ages of one and three, he could predict her height (in inches) by taking her age in months, adding 75 inches and multiplying the result by one-third. Use this information to answer the following:

a) *Write an equation linking her predicted height, h, with her age in months, m.*

b) *Determine her predicted height on her second birthday.*

c) *Determine at what age she is predicted to reach three feet tall.*

Solution

a) To convert the text to an equation, first determine the type of equation we have. We are going to have an equation that links **two variables**. Our unknown will change, depending on the information we are given. For example, we could solve for height given age, or solve for age given height. However, the text gives us a way to determine **height**. Our equation will start with "$h =$".

The text tells us that we can predict her height by taking her age in months, adding 75, and multiplying by $\frac{1}{3}$. So our equation is $h = (m+75) \cdot \frac{1}{3}$, or $h = \frac{1}{3}(m+75)$.

b) To predict Asia's height on her second birthday, we substitute $m = 24$ into our equation (because 2 years is 24 months) and solve for h.

$$\begin{aligned}
h &= \frac{1}{3}(24+75)\\
h &= \frac{1}{3}(99)\\
h &= 33
\end{aligned}$$

Asia's height on her second birthday was predicted to be 33 inches.

c) To determine the predicted age when she reached three feet, substitute $h = 36$ into the equation and solve for m.

$$\begin{aligned}
36 &= \frac{1}{3}(m+75)\\
108 &= m+75\\
33 &= m
\end{aligned}$$

Asia was predicted to be 33 months old when her height was three feet.

Example 9

To convert temperatures in Fahrenheit to temperatures in Celsius, follow the following steps: Take the temperature in degrees Fahrenheit and subtract 32. Then divide the result by 1.8 and this gives the temperature in degrees Celsius.

a) *Write an equation that shows the conversion process.*

b) *Convert 50 degrees Fahrenheit to degrees Celsius.*

c) *Convert 25 degrees Celsius to degrees Fahrenheit.*

d) *Convert -40 degrees Celsius to degrees Fahrenheit.*

a) The text gives the process to convert Fahrenheit to Celsius. We can write an equation using two variables. We will use f for temperature in Fahrenheit, and c for temperature in Celsius.

First we take the temperature in Fahrenheit and subtract 32.	$f-32$
Then divide by 1.8.	$\frac{f-32}{1.8}$
This equals the temperature in Celsius.	$c=\frac{f-32}{1.8}$

In order to convert from one temperature scale to another, simply substitute in for whichever temperature you know, and solve for the one you don't know.

b) To convert 50 degrees Fahrenheit to degrees Celsius, substitute $f = 50$ into the equation.

$$\begin{aligned}
c &= \frac{50-32}{1.8}\\
c &= \frac{18}{1.8}\\
c &= 10
\end{aligned}$$

50 degrees Fahrenheit is equal to 10 degrees Celsius.

c) To convert 25 degrees Celsius to degrees Fahrenheit, substitute $c = 25$ into the equation:

$$\begin{aligned}
25 &= \frac{f-32}{1.8}\\
45 &= f-32\\
77 &= f
\end{aligned}$$

25 degrees Celsius is equal to 77 degrees Fahrenheit.

d) To convert -40 degrees Celsius to degrees Fahrenheit, substitute $c = -40$ into the equation.

$$\begin{aligned}
-40 &= \frac{f-32}{1.8}\\
-72 &= f-32\\
-40 &= f
\end{aligned}$$

-40 degrees Celsius is equal to -40 degrees Fahrenheit. (No, that's not a mistake! This is the one temperature where they are equal.)

Lesson Summary

- Some equations require more than one operation to solve. Generally it, is good to go from the outside in. If there are parentheses around an expression with a variable in it, cancel what is outside the parentheses first.
- Terms with the same variable in them (or no variable in them) are **like terms. Combine like terms** (adding or subtracting them from each other) to simplify the expression and solve for the unknown.

Review Questions

1. Solve the following equations for the unknown variable.
   a. $1.3x - 0.7x = 12$
   b. $6x - 1.3 = 3.2$
   c. $5x - (3x + 2) = 1$
   d. $4(x + 3) = 1$
   e. $5q - 7 = \frac{2}{3}$
   f. $\frac{3}{5}x + \frac{5}{2} = \frac{2}{3}$
   g. $s - \frac{3s}{8} = \frac{5}{6}$
   h. $0.1y + 11 = 0$
   i. $\frac{5q-7}{12} = \frac{2}{3}$
   j. $\frac{5(q-7)}{12} = \frac{2}{3}$
   k. $33t - 99 = 0$
   l. $5p - 2 = 32$
   m. $10y + 5 = 10$
   n. $10(y + 5) = 10$
   o. $10y + 5y = 10$
   p. $10(y + 5y) = 10$
2. Jade is stranded downtown with only \$10 to get home. Taxis cost \$0.75 per mile, but there is an additional \$2.35 hire charge. Write a formula and use it to calculate how many miles she can travel with her money.
3. Jasmin's Dad is planning a surprise birthday party for her. He will hire a bouncy castle, and will provide party food for all the guests. The bouncy castle costs \$150 for the afternoon, and the food will cost \$3 per person. Andrew, Jasmin's Dad, has a budget of \$300. Write an equation and use it to determine the maximum number of guests he can invite.
4. The local amusement park sells summer memberships for \$50 each. Normal admission to the park costs \$25; admission for members costs \$15.
   a. If Darren wants to spend no more than \$100 on trips to the amusement park this summer, how many visits can he make if he buys a membership with part of that money?
   b. How many visits can he make if he does not?
   c. If he increases his budget to \$160, how many visits can he make as a member?
   d. And how many as a non-member?
5. For an upcoming school field trip, there must be one adult supervisor for every five children.
   a. If the bus seats 40 people, how many children can go on the trip?
   b. How many children can go if a second 40-person bus is added?
   c. Four of the adult chaperones decide to arrive separately by car. Now how many children can go in the two buses?

2.6 Equations with Variables on Both Sides

Learning Objectives

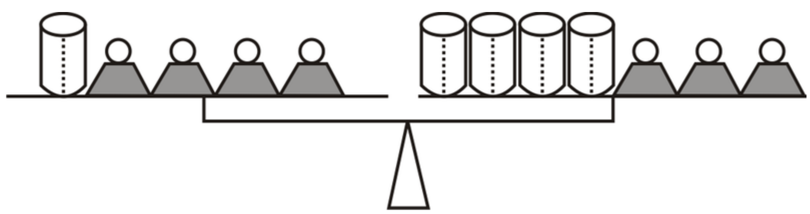
- Solve an equation with variables on both sides.
- Solve an equation with grouping symbols.
- Solve real-world problems using equations with variables on both sides.

Solve an Equation with Variables on Both Sides

When a variable appears on both sides of the equation, we need to manipulate the equation so that all variable terms appear on one side, and only constants are left on the other.

Example 1

Dwayne was told by his chemistry teacher to measure the weight of an empty beaker using a balance. Dwayne found only one lb weights, and so devised the following way of balancing the scales.



Knowing that each weight is one lb, calculate the weight of one beaker.

Solution

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this fact. The unknown quantity, the weight of the beaker, will be our x. We can see that on the left hand scale we have one beaker and four weights. On the right scale, we have four beakers and three weights. The balancing of the scales is analogous to the balancing of the following equation:

$$x+4=4x+3$$

"One beaker plus 4 lbs **equals** 4 beakers plus 3 lbs"

To solve for the weight of the beaker, we want all the constants (numbers) on one side and all the variables (terms with x in them) on the other side. Since there are more beakers on the right and more weights on the left, we'll try to move all the x terms (beakers) to the right, and the constants (weights) to the left.

First we subtract 3 from both sides to get $x+1=4x$.

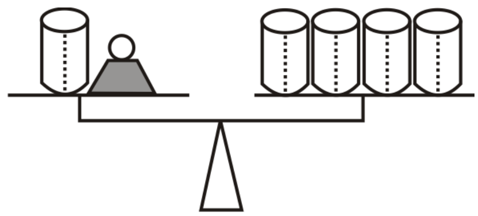
Then we subtract x from both sides to get $1=3x$.

Finally we divide by 3 to get $\frac{1}{3}=x$.

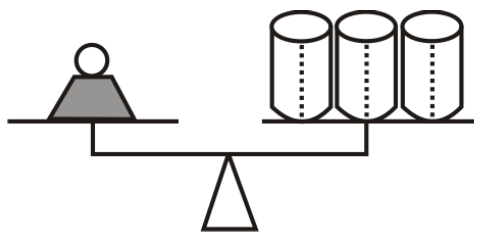
The weight of the beaker is one-third of a pound.

We can do the same with the real objects as we did with the equation. Just as we subtracted amounts from each side of the equation, we could remove a certain number of weights or beakers from each scale. Because we remove the same number of objects from each side, we know the scales will still balance.

First, we could remove three weights from each scale. This would leave one beaker and one weight on the left and four beakers on the right (in other words $x+1=4x$):



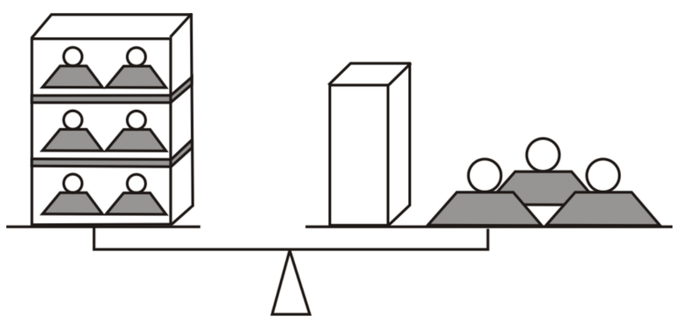
Then we could remove one beaker from each scale, leaving only one weight on the left and three beakers on the right, to get $1=3x$:



Looking at the balance, it is clear that the weight of one beaker is one-third of a pound.

Example 2

Sven was told to find the weight of an empty box with a balance. Sven found some one lb weights and five lb weights. He placed two one lb weights in three of the boxes and with a fourth empty box found the following way of balancing the scales:



Knowing that small weights are one lb and big weights are five lbs, calculate the weight of one box.

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this equality. The unknown quantity—the weight of each empty box, in pounds—will be our x. A box with two 1 lb weights in it weighs $(x+2)$ pounds. Our equation, based on the picture, is $3(x+2)=x+3(5)$.

Distributing the 3 and simplifying, we get $3x+6=x+15$.

Subtracting x from both sides, we get $2x+6=15$.

Subtracting 6 from both sides, we get $2x=9$.

And finally we can divide by 2 to get $x=\frac{9}{2}$, or $x=4.5$.

Each box weighs 4.5 lbs.

Solve an Equation with Grouping Symbols

As you've seen, we can solve equations with variables on both sides even when some of the variables are in parentheses; we just have to get rid of the parentheses, and then we can start combining like terms. We use the same technique when dealing with fractions: first we multiply to get rid of the fractions, and then we can shuffle the terms around by adding and subtracting.

Example 3

Solve $3x+2 = \frac{5x}{3}$.

Solution

The first thing we'll do is get rid of the fraction. We can do this by multiplying both sides by 3, leaving $3(3x+2) = 5x$.

Then we distribute to get rid of the parentheses, leaving $9x+6 = 5x$.

We've already got all the constants on the left side, so we'll move the variables to the right side by subtracting $9x$ from both sides. That leaves us with $6 = -4x$.

And finally, we divide by -4 to get $-\frac{3}{2} = x$, or $x = -1.5$.

Example 4

Solve $7x+2 = \frac{5x-3}{6}$.

Solution

Again we start by eliminating the fraction. Multiplying both sides by 6 gives us $6(7x+2) = 5x-3$, and distributing gives us $42x+12 = 5x-3$.

Subtracting $5x$ from both sides gives us $37x+12 = -3$.

Subtracting 12 from both sides gives us $37x = -15$.

Finally, dividing by 37 gives us $x = -\frac{15}{37}$.

Example 5

Solve the following equation for x: $\frac{14x}{(x+3)} = 7$

Solution

The form of the left hand side of this equation is known as a **rational function** because it is the ratio of two other functions: $14x$ and $(x+3)$. But we can solve it just like any other equation involving fractions.

First we multiply both sides by $(x+3)$ to get rid of the fraction. Now our equation is $14x = 7(x+3)$.

Then we distribute: $14x = 7x+21$.

Then subtract $7x$ from both sides: $7x = 21$.

And divide by 7: $x = 3$.

Solve Real-World Problems Using Equations with Variables on Both Sides

Here's another chance to practice translating problems from words to equations. What is the equation asking? What is the **unknown** variable? What quantity will we use for our variable?

The text explains what's happening. Break it down into small, manageable chunks, and follow what's going on with our variable all the way through the problem.

More on Ohm's Law

Recall that the electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

The resistance R of a number of components wired in a **series** (one after the other) is simply the sum of all the resistances of the individual components.

Example 6

In an attempt to find the resistance of a new component, a scientist tests it in series with standard resistors. A fixed voltage causes a 4.8 amp current in a circuit made up from the new component plus a 15Ω *resistor in series. When the component is placed in a series circuit with a* 50Ω *resistor, the same voltage causes a 2.0 amp current to flow. Calculate the resistance of the new component.*

This is a complex problem to translate, but once we convert the information into equations it's relatively straightforward to solve. First, we are trying to find the resistance of the new component (in Ohms, Ω). This is our x. We don't know the voltage that is being used, but we can leave that as a variable, V. Our first situation has a total resistance that equals the unknown resistance plus 15Ω. The current is 4.8 amps. Substituting into the formula $V = I \cdot R$, we get $V = 4.8(x+15)$.

Our second situation has a total resistance that equals the unknown resistance plus 50Ω. The current is 2.0 amps. Substituting into the same equation, this time we get $V = 2(x+50)$.

We know the voltage is fixed, so the V in the first equation must equal the V in the second. That means we can set the right-hand sides of the two equations equal to each other: $4.8(x+15) = 2(x+50)$. Then we can solve for x.

Distribute the constants first: $4.8x+72 = 2x+100$.

Subtract $2x$ from both sides: $2.8x+72 = 100$.

Subtract 72 from both sides: $2.8x = 28$.

Divide by 2.8: $x = 10$.

The resistance of the component is 10Ω**.**

Lesson Summary

If an unknown variable appears on both sides of an equation, distribute as necessary. Then simplify the equation to have the unknown on only one side.

Review Questions

1. Solve the following equations for the unknown variable.
   a. $3(x-1) = 2(x+3)$
   b. $7(x+20) = x+5$
   c. $9(x-2) = 3x+3$
   d. $2\left(a-\frac{1}{3}\right) = \frac{2}{5}\left(a+\frac{2}{3}\right)$
   e. $\frac{2}{7}\left(t+\frac{2}{3}\right) = \frac{1}{5}\left(t-\frac{2}{3}\right)$
   f. $\frac{1}{7}\left(v+\frac{1}{4}\right) = 2\left(\frac{3v}{2}-\frac{5}{2}\right)$
   g. $\frac{y-4}{11} = \frac{2}{5} \cdot \frac{2y+1}{3}$
   h. $\frac{z}{16} = \frac{2(3z+1)}{9}$
   i. $\frac{q}{16} + \frac{q}{6} = \frac{(3q+1)}{9} + \frac{3}{2}$
   j. $\frac{3}{x} = \frac{2}{x+1}$
   k. $\frac{5}{2+p} = \frac{3}{p-8}$
2. Manoj and Tamar are arguing about a number trick they heard. Tamar tells Andrew to think of a number, multiply it by five and subtract three from the result. Then Manoj tells Andrew to think of a number, add five and multiply the result by three. Andrew says that whichever way he does the trick he gets the same answer.
   a. What was the number Andrew started with?

b. What was the result Andrew got both times?
c. Name another set of steps that would have resulted in the same answer if Andrew started with the same number.

3. Manoj and Tamar try to come up with a harder trick. Manoj tells Andrew to think of a number, double it, add six, and then divide the result by two. Tamar tells Andrew to think of a number, add five, triple the result, subtract six, and then divide the result by three.
   a. Andrew tries the trick both ways and gets an answer of 10 each time. What number did he start out with?
   b. He tries again and gets 2 both times. What number did he start out with?
   c. Is there a number Andrew can start with that will *not* give him the same answer both ways?
   d. **Bonus:** Name another set of steps that would give Andrew the same answer every time as he would get from Manoj's and Tamar's steps.
4. I have enough money to buy five regular priced CDs and have \$6 left over. However, all CDs are on sale today, for \$4 less than usual. If I borrow \$2, I can afford nine of them.
   a. How much are CDs on sale for today?
   b. How much would I have to borrow to afford nine of them if they weren't on sale?
5. Five identical electronics components were connected in series. A fixed but unknown voltage placed across them caused a 2.3 amp current to flow. When two of the components were replaced with standard 10Ω resistors, the current dropped to 1.9 amps. What is the resistance of each component?
6. Solve the following resistance problems. Assume the same voltage is applied to all circuits.
   a. Three unknown resistors plus 20Ω give the same current as one unknown resistor plus 70Ω.
   b. One unknown resistor gives a current of 1.5 amps and a 15Ω resistor gives a current of 3.0 amps.
   c. Seven unknown resistors plus 18Ω gives twice the current of two unknown resistors plus 150Ω.
   d. Three unknown resistors plus 1.5Ω gives a current of 3.6 amps and seven unknown resistors plus seven 12Ω resistors gives a current of 0.2 amps.

2.7 Multi-Step Equations

Learning Objectives

- Solve a multi-step equation by combining like terms.
- Solve a multi-step equation using the distributive property.
- Solve real-world problems using multi-step equations.

Solving Multi-Step Equations by Combining Like Terms

We've seen that when we solve for an unknown variable, it can take just one or two steps to get the terms in the right places. Now we'll look at solving equations that take several steps to isolate the unknown variable. Such equations are referred to as **multi-step equations**.

In this section, we'll simply be combining the steps we already know how to do. Our goal is to end up with all the constants on one side of the equation and all the variables on the other side. We'll do this by collecting like terms. Don't forget, like terms have the same combination of variables in them.

Example 1

Solve $\frac{3x+4}{3} - 5x = 6$.

Before we can combine the variable terms, we need to get rid of that fraction.

First let's put all the terms on the left over a common denominator of three: $\frac{3x+4}{3} - \frac{15x}{3} = 6$.

Combining the fractions then gives us $\frac{3x+4-15x}{3} = 6$.

Combining like terms in the numerator gives us $\frac{4-12x}{3} = 6$.

Multiplying both sides by 3 gives us $4 - 12x = 18$.

Subtracting 4 from both sides gives us $-12x = 14$.

And finally, dividing both sides by -12 gives us $x = -\frac{14}{12}$, which reduces to $x = -\frac{7}{6}$.

Solving Multi-Step Equations Using the Distributive Property

You may have noticed that when one side of the equation is multiplied by a constant term, we can either distribute it or just divide it out. If we can divide it out without getting awkward fractions as a result, then that's usually the better choice, because it gives us smaller numbers to work with. But if dividing would result in messy fractions, then it's usually better to distribute the constant and go from there.

Example 2

Solve $7(2x-5) = 21$.

The first thing we want to do here is get rid of the parentheses. We could use the Distributive Property, but it just so happens that 7 divides evenly into 21. That suggests that dividing both sides by 7 is the easiest way to solve this problem.

If we do that, we get $2x - 5 = \frac{21}{7}$ or just $2x - 5 = 3$. Then all we need to do is add 5 to both sides to get $2x = 8$, and then divide by 2 to get $x = 4$.

Example 3

Solve $17(3x+4) = 7$.

Once again, we want to get rid of those parentheses. We could divide both sides by 17, but that would give us an inconvenient fraction on the right-hand side. In this case, distributing is the easier way to go.

Distributing the 17 gives us $51x+68 = 7$. Then we subtract 68 from both sides to get $51x = -61$, and then we divide by 51 to get $x = \frac{-61}{51}$. (Yes, that's a messy fraction too, but since it's our final answer and we don't have to do anything else with it, we don't really care how messy it is.)

Example 4

Solve $4(3x-4)-7(2x+3) = 3$.

Before we can collect like terms, we need to get rid of the parentheses using the Distributive Property. That gives us $12x-16-14x-21 = 3$, which we can rewrite as $(12x-14x)+(-16-21) = 3$. This in turn simplifies to $-2x-37 = 3$.

Next we add 37 to both sides to get $-2x = 40$.

And finally, we divide both sides by -2 to get $x = -20$.

Example 5

Solve the following equation for x: $0.1(3.2+2x)+\frac{1}{2}\left(3-\frac{x}{5}\right) = 0$

This function contains both fractions and decimals. We should convert all terms to one or the other. It's often easier to convert decimals to fractions, but in this equation the fractions are easy to convert to decimals—and with decimals we don't need to find a common denominator!

In decimal form, our equation becomes $0.1(3.2+2x)+0.5(3-0.2x) = 0$.

Distributing to get rid of the parentheses, we get $0.32+0.2x+1.5-0.1x = 0$.

Collecting and combining like terms gives us $0.1x+1.82 = 0$.

Then we can subtract 1.82 from both sides to get $0.1x = -1.82$, and finally divide by 0.1 (or multiply by 10) to get $x = -18.2$.

Solving Real-World Problems Using Multi-Step Equations

Example 6

A growers' cooperative has a farmer's market in the town center every Saturday. They sell what they have grown and split the money into several categories. 8.5% of all the money taken in is set aside for sales tax. $150 goes to pay the rent on the space they occupy. What remains is split evenly between the seven growers. How much total money is taken in if each grower receives a $175 share?

Let's translate the text above into an equation. The unknown is going to be the total money taken in dollars. We'll call this x.

"8.5% of all the money taken in is set aside for sales tax." This means that 91.5% of the money remains. This is $0.915x$.

"$150 goes to pay the rent on the space they occupy." This means that what's left is $0.915x-150$.

"What remains is split evenly between the 7 growers." That means each grower gets $\frac{0.915x-150}{7}$.

If each grower's share is $175, then our equation to find x is $\frac{0.915x-150}{7} = 175$.

First we multiply both sides by 7 to get $0.915x-150 = 1225$.

Then add 150 to both sides to get $0.915x = 1375$.

Finally divide by 0.915 to get $x \approx 1502.7322$. Since we want our answer in dollars and cents, we round to two decimal places, or \$1502.73.

The workers take in a total of \$1502.73.

Example 7

A factory manager is packing engine components into wooden crates to be shipped on a small truck. The truck is designed to hold sixteen crates, and will safely carry a 1200 lb cargo. Each crate weighs 12 lbs empty. How much weight should the manager instruct the workers to put in each crate in order to get the shipment weight as close as possible to 1200 lbs?

The unknown quantity is the weight to put in each box, so we'll call that x.

Each crate when full will weigh $x+12$ *lbs*, so all 16 crates together will weigh $16(x+12)$ *lbs*.

We also know that all 16 crates together should weigh 1200 lbs, so we can say that $16(x+12) = 1200$.

To solve this equation, we can start by dividing both sides by 16: $x+12 = \frac{1200}{16} = 75$.

Then subtract 12 from both sides: $x = 63$.

The manager should tell the workers to put 63 lbs of components in each crate.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

Example 8

A scientist is trying to deduce the resistance of an unknown component. He labels the resistance of the unknown component x Ω. The resistance of a circuit containing a number of these components is $(5x+20)\Omega$. If a 120 volt potential difference across the circuit produces a current of 2.5 amps, calculate the resistance of the unknown component.

Solution

To solve this, we need to start with the equation $V = I \cdot R$ and substitute in $V = 120, I = 2.5$, and $R = 5x+20$. That gives us $120 = 2.5(5x+20)$.

Distribute the 2.5 to get $120 = 12.5x+50$.

Subtract 50 from both sides to get $70 = 12.5x$.

Finally, divide by 12.5 to get $5.6 = x$.

The unknown components have a resistance of 5.6 Ω.

Distance, Speed and Time

The speed of a body is the distance it travels per unit of time. That means that we can also find out how far an object moves in a certain amount of time if we know its speed: we use the equation "distance $=$ speed $\times$ time."

Example 8

Shanice's car is traveling 10 miles per hour slower than twice the speed of Brandon's car. She covers 93 miles in 1 hour 30 minutes. How fast is Brandon driving?

Solution

Here, we don't know either Brandon's speed or Shanice's, but since the question asks for Brandon's speed, that's what we'll use as our variable x.

The distance Shanice covers in miles is 93, and the time in hours is 1.5. Her speed is 10 less than twice Brandon's speed, or $2x-10$ miles per hour. Putting those numbers into the equation gives us $93 = 1.5(2x-10)$.

First we distribute, to get $93 = 3x-15$.

Then we add 15 to both sides to get $108 = 3x$.

Finally we divide by 3 to get $36 = x$.

Brandon is driving at 36 miles per hour.

We can check this answer by considering the situation another way: we can solve for Shanice's speed instead of Brandon's and then check that against Brandon's speed. We'll use y for Shanice's speed since we already used x for Brandon's.

The equation for Shanice's speed is simply $93 = 1.5y$. We can divide both sides by 1.5 to get $62 = y$, so Shanice is traveling at 62 miles per hour.

The problem tells us that Shanice is traveling 10 mph slower than twice Brandon's speed; that would mean that 62 is equal to 2 times 36 minus 10. Is that true? Well, 2 times 36 is 72, minus 10 is 62. The answer checks out.

In algebra, there's almost always more than one method of solving a problem. If time allows, it's always a good idea to try to solve the problem using two different methods just to confirm that you've got the answer right.

Speed of Sound

The speed of sound in dry air, v, is given by the equation $v = 331+0.6T$, where T is the temperature in Celsius and v is the speed of sound in meters per second.

Example 9

Tashi hits a drainpipe with a hammer and 250 meters away Minh hears the sound and hits his own drainpipe. Unfortunately, there is a one second delay between him hearing the sound and hitting his own pipe. Tashi accurately measures the time between her hitting the pipe and hearing Mihn's pipe at 2.46 seconds. What is the temperature of the air?

This is a complex problem and we need to be careful in writing our equations. First of all, the distance the sound travels is equal to the speed of sound multiplied by the time, and the speed is given by the equation above. So the distance equals $(331+0.6T) \times \text{time}$, and the time is $2.46-1$ (because for 1 second out of the 2.46 seconds measured, there was no sound actually traveling). We also know that the distance is 250×2 (because the sound traveled from Tashi to Minh and back again), so our equation is $250 \times 2 = (331+0.6T)(2.46-1)$, which simplifies to $500 = 1.46(331+0.6T)$.

Distributing gives us $500 = 483.26+0.876T$, and subtracting 483.26 from both sides gives us $16.74 = 0.876T$. Then we divide by 0.876 to get $T \approx 19.1$.

The temperature is about 19.1 degrees Celsius.

Lesson Summary

- Multi-step equations are slightly more complex than one - and two-step equations, but use the same basic techniques.
- If dividing a number outside of parentheses will produce fractions, it is often better to use the **Distributive Property** to expand the terms and then combine like terms to solve the equation.

Review Questions

1. Solve the following equations for the unknown variable.

a. $3(x-1)-2(x+3)=0$
b. $3(x+3)-2(x-1)=0$
c. $7(w+20)-w=5$
d. $5(w+20)-10w=5$
e. $9(x-2)-3x=3$
f. $12(t-5)+5=0$
g. $2(2d+1)=\frac{2}{3}$
h. $2\left(5a-\frac{1}{3}\right)=\frac{2}{7}$
i. $\frac{2}{9}\left(i+\frac{2}{3}\right)=\frac{2}{5}$
j. $4\left(v+\frac{1}{4}\right)=\frac{35}{2}$
k. $\frac{g}{10}=\frac{6}{3}$
l. $\frac{s-4}{11}=\frac{2}{5}$
m. $\frac{2k}{7}=\frac{3}{8}$
n. $\frac{7x+4}{3}=\frac{9}{2}$
o. $\frac{9y-3}{6}=\frac{5}{2}$
p. $\frac{r}{3}+\frac{r}{2}=7$
q. $\frac{p}{16}-\frac{2p}{3}=\frac{1}{9}$
r. $\frac{m+3}{2}-\frac{m}{4}=\frac{1}{3}$
s. $5\left(\frac{k}{3}+2\right)=\frac{32}{3}$
t. $\frac{3}{z}=\frac{2}{5}$
u. $\frac{2}{r}+2=\frac{10}{3}$
v. $\frac{12}{5}=\frac{3+z}{z}$

2. An engineer is building a suspended platform to raise bags of cement. The platform has a mass of 200 kg, and each bag of cement is 40 kg. He is using two steel cables, each capable of holding 250 kg. Write an equation for the number of bags he can put on the platform at once, and solve it.
3. A scientist is testing a number of identical components of unknown resistance which he labels $x\Omega$. He connects a circuit with resistance $(3x+4)\Omega$ to a steady 12 volt supply and finds that this produces a current of 1.2 amps. What is the value of the unknown resistance?
4. Lydia inherited a sum of money. She split it into five equal parts. She invested three parts of the money in a high-interest bank account which added 10% to the value. She placed the rest of her inheritance plus \$500 in the stock market but lost 20% on that money. If the two accounts end up with exactly the same amount of money in them, how much did she inherit?
5. Pang drove to his mother's house to drop off her new TV. He drove at 50 miles per hour there and back, and spent 10 minutes dropping off the TV. The entire journey took him 94 minutes. How far away does his mother live?

2.8 Rational Equations Using Proportions

Here you'll learn how to use proportions to find the solutions to rational equations.

Solution of Rational Equations

You are now ready to solve rational equations! There are two main methods you will learn to solve rational equations:

- Cross products
- Lowest common denominators

In this Concept you will learn how to solve using cross products.

Solving a Rational Proportion

When two rational expressions are equal, a **proportion** is created and can be solved using its cross products.

Example

For example, to solve $\frac{x}{5} = \frac{(x+1)}{2}$, cross multiply and the products are equal.

$$\frac{x}{5} = \frac{(x+1)}{2} \rightarrow 2(x) = 5(x+1)$$

Solve for x:

$$\begin{aligned} 2(x) = 5(x+1) &\rightarrow 2x = 5x+5 \\ 2x-5x &= 5x-5x+5 \\ -3x &= 5 \\ x &= -\frac{5}{3} \end{aligned}$$

Solve the following equations.

1. $\frac{2x+1}{4} = \frac{x-3}{10}$
2. $\frac{4x}{x+2} = \frac{5}{9}$
3. $\frac{5}{3x-4} = \frac{2}{x+1}$
4. $\frac{2}{x+3} - \frac{1}{x+4} = 0$

Mixed Review

7. Divide: $-2\frac{9}{10} \div -\frac{15}{8}$.
8. Solve for $g : -1.5\left(-3\frac{4}{5}+g\right)=\frac{201}{20}$.
9. Find the discriminant of $6x^2+3x+4=0$ and determine the nature of the roots.
10. Simplify $\frac{6b}{2b+2}+3$.
11. Simplify $\frac{8}{2x-4}-\frac{5x}{x-5}$.
12. Divide: $(7x^2+16x-10) \div (x+3)$.
13. Simplify $(n-1)*(3n+2)(n-4)$.

2.9 Solving Rational Equations using the LCD

Here you'll use the LCD of the expressions in a rational equation in order to solve for x.

A right triangle has leg lengths of $\frac{1}{2}$ and $\frac{1}{x}$ units. Its hypotenuse is 2 units. What is the value of x.

Guidance

In addition to using cross-multiplication to solve a rational equation, we can also use the LCD of all the rational expressions within the equation and eliminate the fraction. To demonstrate, we will walk through a few examples.

Example A

Solve $\frac{5}{2}+\frac{1}{x}=3$.

Solution: The LCD for 2 and x is $2x$. Multiply each term by $2x$, so that the denominators are eliminated. We will put the $2x$ over 1, when multiplying it by the fractions, so that it is easier to line up and cross-cancel.

$$\begin{aligned}\frac{5}{2}+\frac{1}{x}&=3\\ \frac{\cancel{2}x}{1}\cdot\frac{5}{\cancel{2}}+\frac{2\cancel{x}}{1}\cdot\frac{1}{\cancel{x}}&=2x\cdot 3\\ 5x+2&=6x\\ 2&=x\end{aligned}$$

Checking the answer, we have $\frac{5}{2}+\frac{1}{2}=3\rightarrow\frac{6}{2}=3$

Example B

Solve $\frac{5x}{x-2}=7+\frac{10}{x-2}$.

Solution: Because the denominators are the same, we need to multiply all three terms by $x-2$.

$$\begin{aligned}\frac{5x}{x-2}&=7+\frac{10}{x-2}\\ \cancel{(x-2)}\cdot\frac{5x}{\cancel{x-2}}&=(x-2)\cdot 7+\cancel{(x-2)}\cdot\frac{10}{\cancel{x-2}}\\ 5x&=7x-14+10\\ -2x&=-4\\ x&=2\end{aligned}$$

Checking our answer, we have: $\frac{5\cdot 2}{2-2}=7+\frac{10}{2-2}\rightarrow\frac{10}{0}=7+\frac{10}{0}$. Because the solution is the vertical asymptote of two of the expressions, $x=2$ is an extraneous solution. Therefore, there is no solution to this problem.

Example C

Solve $\frac{3}{x}+\frac{4}{5}=\frac{6}{x-2}$.

Solution: Determine the LCD for 5, x, and $x-2$. It would be the three numbers multiplied together: $5x(x-2)$. Multiply each term by the LCD.

$$\frac{3}{x}+\frac{4}{5}=\frac{6}{x-2}$$
$$\frac{5\cancel{x}(x-2)}{1}\cdot\frac{3}{\cancel{x}}+\frac{\cancel{5}x(x-2)}{1}\cdot\frac{4}{\cancel{5}}=\frac{5x\cancel{(x-2)}}{1}\cdot\frac{6}{\cancel{x-2}}$$
$$15(x-2)+4x(x-2)=30x$$

Multiplying each term by the entire LCD cancels out each denominator, so that we have an equation that we have learned how to solve in previous concepts. Distribute the 15 and $4x$, combine like terms and solve.

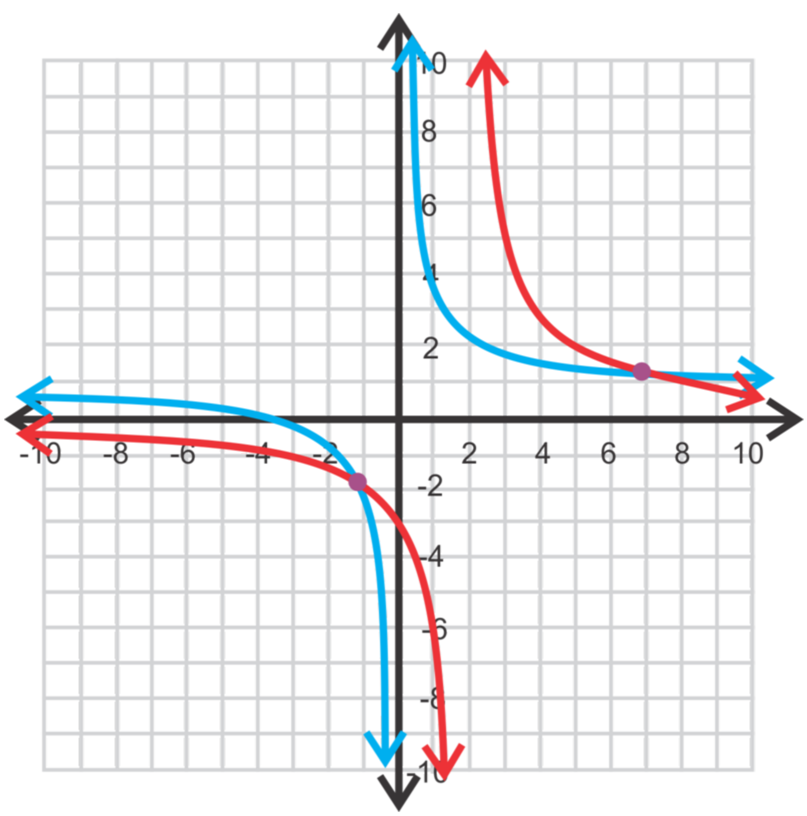
$$15x-30+4x^2-8x=30x$$
$$4x^2-23x-30=0$$

This polynomial is not factorable. Let's use the Quadratic Formula to find the solutions.

$x=\frac{23\pm\sqrt{(-23)^2-4\cdot4\cdot(-30)}}{2\cdot4}=\frac{23\pm\sqrt{1009}}{8}$

Approximately, the solutions are $\frac{23+\sqrt{1009}}{8}\approx 6.85$ and $\frac{23-\sqrt{1009}}{8}\approx -1.096$. It is harder to check these solutions. The easiest thing to do is to graph $\frac{3}{x}+\frac{4}{5}$ in $Y1$ and $\frac{6}{x-2}$ in $Y2$ (using your graphing calculator).

The x-values of the points of intersection (purple points in the graph) are approximately the same as the solutions we found.



Intro Problem Revisit We need to use the Pythagorean Theorem to solve for x.

$$
\begin{aligned}
(\frac{1}{2})^2+(\frac{1}{x})^2 &= 2^2 \\
\frac{1}{4}+\frac{1}{x^2} &= 4 \\
\frac{\cancel{4}x^2}{1}\cdot\frac{1}{\cancel{4}}+\frac{4\cancel{x^2}}{1}\cdot\frac{1}{\cancel{x^2}} &= 4\cdot 4x^2 \\
x^2+4 &= 16x^2 \\
4 &= 15x^2 \\
\frac{4}{15} &= x^2 \\
x &= \frac{2\sqrt{15}}{15}
\end{aligned}
$$

Guided Practice

Solve the following equations.

1. $\frac{2x}{x-3}=2+\frac{3x}{x^2-9}$
2. $\frac{4}{x-3}+5=\frac{9}{x+2}$
3. $\frac{3}{x^2+4x+4}+\frac{1}{x+2}=\frac{2}{x^2-4}$

Answers

1. The LCD is x^2-9. Multiply each term by its factored form to cross-cancel.

$$
\begin{aligned}
\frac{2x}{x-3} &= 2+\frac{3x}{x^2-9} \\
\frac{\cancel{(x-3)}(x+3)}{1}\cdot\frac{2x}{\cancel{x-3}} &= (x-3)(x+3)\cdot 2+\frac{\cancel{(x-3)(x+3)}}{1}\cdot\frac{3x}{\cancel{x^2-9}} \\
2x(x+3) &= 2(x^2-9)+3x \\
2x^2+6x &= 2x^2-18+3x \\
3x &= -18 \\
x &= -6
\end{aligned}
$$

Checking our answer, we have: $\frac{2(-6)}{-6-3}=2+\frac{3(-6)}{(-6)^2-9}\rightarrow\frac{-12}{-9}=2+\frac{-18}{27}\rightarrow\frac{4}{3}=2-\frac{2}{3}$

2. The LCD is $(x-3)(x+2)$. Multiply each term by the LCD.

$$\frac{4}{x-3}+5=\frac{9}{x+2}$$
$$\cancel{(x-3)}(x+2)\cdot\frac{4}{\cancel{x-3}}+(x-3)(x+2)\cdot 5=(x-3)\cancel{(x+2)}\cdot\frac{9}{\cancel{x+2}}$$
$$4(x+2)+5(x-3)(x+2)=9(x-3)$$
$$4x+8+5x^2-5x-30=9x-27$$
$$5x^2-10x+5=0$$
$$5(x^2-2x+1)=0$$

This polynomial factors to be $5(x-1)(x-1)=0$, so $x=1$ is a repeated solution. Checking our answer, we have $\frac{4}{1-3}+5=\frac{9}{1+2}\rightarrow -2+5=3$

3. The LCD is $(x+2)(x+2)(x-2)$.

$$\frac{3}{x^2+4x+4}+\frac{1}{x+2}=\frac{2}{x^2-4}$$
$$\cancel{(x+2)(x+2)}(x-2)\cdot\frac{3}{\cancel{(x+2)(x+2)}}+\cancel{(x+2)}(x+2)(x-2)\cdot\frac{1}{\cancel{x+2}}=(x+2)\cancel{(x+2)(x-2)}\cdot\frac{2}{\cancel{(x-2)(x+2)}}$$
$$3(x-2)+(x-2)(x+2)=2(x+2)$$
$$3x-6+x^2-4=2x+4$$
$$x^2+x-14=0$$

This quadratic is not factorable, so we need to use the Quadratic Formula to solve for x.

$x=\frac{-1\pm\sqrt{1-4(-14)}}{2}=\frac{-1\pm\sqrt{57}}{2}\approx 3.27$ and -4.27

Using your graphing calculator, you can check the answer. The x-values of points of intersection of $y=\frac{3}{x^2+4x+4}+\frac{1}{x+2}$ and $y=\frac{2}{x^2-4}$ are the same as the values above.

Practice

Determine if the following values for x are solutions for the given equations.

1. $\frac{4}{x-3}+2=\frac{3}{x+4}$, $x=-1$
2. $\frac{2x-1}{x-5}-3=\frac{x+6}{2x}$, $x=6$

What is the LCD for each set of numbers?

3. $4-x,\ x^2-16$
4. $2x,\ 6x-12,\ x^2-9$
5. $x-3,\ x^2-x-6,\ x^2-4$

Solve the following equations.

6. $\frac{6}{x+2}+1=\frac{5}{x}$
7. $\frac{5}{3x}-\frac{2}{x+1}=\frac{4}{x}$

8. $\frac{12}{x^2-9} = \frac{8x}{x-3} - \frac{2}{x+3}$
9. $\frac{6x}{x^2-1} + \frac{2}{x+1} = \frac{3x}{x-1}$
10. $\frac{5x-3}{4x} - \frac{x+1}{x+2} = \frac{1}{x^2+2x}$
11. $\frac{4x}{x^2+6x+9} - \frac{2}{x+3} = \frac{3}{x^2-9}$
12. $\frac{x^2}{x^2-8x+16} = \frac{x}{x-4} + \frac{3x}{x^2-16}$
13. $\frac{5x}{2x-3} + \frac{x+1}{x} = \frac{6x^2+x+12}{2x^2-3x}$
14. $\frac{3x}{x^2+2x-8} = \frac{x+1}{x^2+4x} + \frac{2x+1}{x^2-2x}$
15. $\frac{x+1}{x^2+7x} + \frac{x+2}{x^2-3x} = \frac{x}{x^2+4x-21}$

2.10 Radical Equations

Here you'll learn how to find the solutions to radical equations.

Guidance

Solving radical equations is no different from solving linear or quadratic equations. Before you can begin to solve a radical equation, you must know how to cancel the radical. To do that, you must know its **inverse**.

TABLE 2.9:

Original Operation	**Inverse Operation**
Cube Root	Cubing (to the third power)
Square Root	Squaring (to the second power)
Fourth Root	Fourth power
"nth" Root	"nth" power

To solve a radical equation, you apply the solving equation steps you learned in previous Concepts, including the inverse operations for roots.

Example A

Solve $\sqrt{2x-1}=5$.

Solution:

The first operation that must be removed is the square root. Square both sides.

$$\begin{aligned}\left(\sqrt{2x-1}\right)^2 &= 5^2\\ 2x-1 &= 25\\ 2x &= 26\\ x &= 13\end{aligned}$$

Remember to check your answer by substituting it into the original problem to see if it makes sense.

Extraneous Solutions

Not every solution of a radical equation will check in the original problem. This is called an **extraneous solution**. This means you can find a solution using algebra, but it will not work when checked. This is because of the rule in a previous Concept:

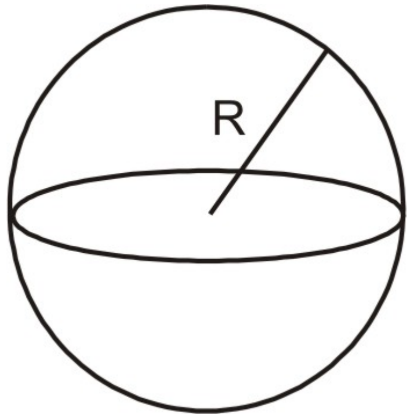
$\sqrt[n]{x}$ is undefined when n is an even whole number and $x<0$,

or, in words, *even roots of negative numbers are undefined.*

Radical Equations in Real Life

Example B

A sphere has a volume of $456\ cm^3$. *If the radius of the sphere is increased by 2 cm, what is the new volume of the sphere?*



Solution:

1. **Define variables.** Let $R =$ the radius of the sphere.
2. **Find an equation.** The volume of a sphere is given by the formula: $V = \frac{4}{3}\pi r^3$.

By substituting 456 for the volume variable, the equation becomes $456 = \frac{4}{3}\pi r^3$.

$$\begin{aligned}
\text{Multiply by 3 :} \quad & 1368 = 4\pi r^3 \\
\text{Divide by } 4\pi : \quad & 108.92 = r^3 \\
\text{Take the cube root of each side:} \quad & r = \sqrt[3]{108.92} \Rightarrow r = 4.776\ cm \\
\text{The new radius is 2 centimeters more:} \quad & r = 6.776\ cm \\
\text{The new volume is :} \quad & V = \frac{4}{3}\pi(6.776)^3 = 1302.5\ cm^3
\end{aligned}$$

Check by substituting the values of the radii into the volume formula.

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4.776)^3 = 456\ cm^3$. The solution checks out.

Guided Practice

Solve $\sqrt{x+15} = \sqrt{3x-3}$.

Solution:

Begin by canceling the square roots by squaring both sides.

$$\begin{aligned}
\left(\sqrt{x+15}\right)^2 &= \left(\sqrt{3x-3}\right)^2 \\
x+15 &= 3x-3 \\
\text{Isolate the } x-\text{variable :} \quad 18 &= 2x \\
x &= 9
\end{aligned}$$

Check the solution: $\sqrt{9+15} = \sqrt{3(9)-3} \rightarrow \sqrt{24} = \sqrt{24}$. The solution checks.

Practice

In 1-16, find the solution to each of the following radical equations. Identify extraneous solutions.

1. $\sqrt{x+2}-2=0$
2. $\sqrt{3x-1}=5$
3. $2\sqrt{4-3x}+3=0$
4. $\sqrt[3]{x-3}=1$
5. $\sqrt[4]{x^2-9}=2$
6. $\sqrt[3]{-2-5x}+3=0$
7. $\sqrt{x^2-5x}-6=0$
8. $\sqrt{3x+4}=-6$
9. The area of a triangle is 24 in^2 and the height of the triangle is twice as long and the base. What are the base and the height of the triangle?
10. The volume of a square pyramid is given by the formula $V=\frac{A(h)}{3}$, where $A=$ *area of the base* and $h=$ *height of the pyramid*. The volume of a square pyramid is 1,600 cubic meters. If its height is 10 meters, find the area of its base.
11. The volume of a cylinder is 245 cm^3 and the height of the cylinder is one-third the diameter of the cylinder's base. The diameter of the cylinder is kept the same, but the height of the cylinder is increased by two centimeters. What is the volume of the new cylinder? $(\text{Volume} = \pi r^2 \cdot h)$
12. The height of a golf ball as it travels through the air is given by the equation $h=-16t^2+256$. Find the time when the ball is at a height of 120 feet.

Mixed Review

21. Joy sells two types of yarn: wool and synthetic. Wool is \$12 per skein and synthetic is \$9 per skein. If Joy sold 16 skeins of synthetic and collected a total of \$432, how many skeins of wool did she sell?
22. Solve $16 \geq |x-4|$.
23. Graph the solution: $\begin{cases} y \leq 2x-4 \\ y > -\frac{1}{4}x+6 \end{cases}$.
24. You randomly point to a day in the month of February, 2011. What is the probability your finger lands on a Monday?
25. Carbon-14 has a half life of 5,730 years. Your dog dug a bone from your yard. It had 93% of its carbon-14 remaining. How old is the bone?
26. What is true about solutions to inconsistent systems?

2.11 Solving Real-World Problems Using Multi-Step Equations

Here you'll learn how to translate words into to multi-step equations. You'll then solve such equations for their unknown variable.

Guidance

We can now use strategies for solving multi-step equations to solve real world equations.

Example A

A growers' cooperative has a farmer's market in the town center every Saturday. They sell what they have grown and split the money into several categories. 8.5% of all the money taken in is set aside for sales tax. $150 goes to pay the rent on the space they occupy. What remains is split evenly between the seven growers. How much total money is taken in if each grower receives a $175 share?

Let's translate the text above into an equation. The unknown is going to be the total money taken in dollars. We'll call this x.

"8.5% of all the money taken in is set aside for sales tax." This means that 91.5% of the money remains. This is $0.915x$.

"$150 goes to pay the rent on the space they occupy." This means that what's left is $0.915x - 150$.

"What remains is split evenly between the 7 growers." That means each grower gets $\frac{0.915x-150}{7}$.

If each grower's share is $175, then our equation to find x is $\frac{0.915x-150}{7} = 175$.

First we multiply both sides by 7 to get $0.915x - 150 = 1225$.

Then add 150 to both sides to get $0.915x = 1375$.

Finally divide by 0.915 to get $x \approx 1502.7322$. Since we want our answer in dollars and cents, we round to two decimal places, or $1502.73.

The workers take in a total of $1502.73.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

Example B

A scientist is trying to deduce the resistance of an unknown component. He labels the resistance of the unknown component x Ω. The resistance of a circuit containing a number of these components is $(5x+20)\Omega$. If a 120 volt potential difference across the circuit produces a current of 2.5 amps, calculate the resistance of the unknown component.

Solution

To solve this, we need to start with the equation $V = I \cdot R$ and substitute in $V = 120, I = 2.5$, and $R = 5x+20$. That gives us $120 = 2.5(5x+20)$.

Distribute the 2.5 to get $120 = 12.5x + 50$.

Subtract 50 from both sides to get $70 = 12.5x$.

Finally, divide by 12.5 to get $5.6 = x$.

The unknown components have a resistance of $5.6\ \Omega$**.**

Distance, Speed and Time

The speed of a body is the distance it travels per unit of time. That means that we can also find out how far an object moves in a certain amount of time if we know its speed: we use the equation "distance = speed × time."

Example C

Shanice's car is traveling 10 miles per hour slower than twice the speed of Brandon's car. She covers 93 miles in 1 hour 30 minutes. How fast is Brandon driving?

Solution

Here, we don't know either Brandon's speed or Shanice's, but since the question asks for Brandon's speed, that's what we'll use as our variable x.

The distance Shanice covers in miles is 93, and the time in hours is 1.5. Her speed is 10 less than twice Brandon's speed, or $2x - 10$ miles per hour. Putting those numbers into the equation gives us $93 = 1.5(2x - 10)$.

First we distribute, to get $93 = 3x - 15$.

Then we add 15 to both sides to get $108 = 3x$.

Finally we divide by 3 to get $36 = x$.

Brandon is driving at 36 miles per hour.

We can check this answer by considering the situation another way: we can solve for Shanice's speed instead of Brandon's and then check that against Brandon's speed. We'll use y for Shanice's speed since we already used x for Brandon's.

The equation for Shanice's speed is simply $93 = 1.5y$. We can divide both sides by 1.5 to get $62 = y$, so Shanice is traveling at 62 miles per hour.

The problem tells us that Shanice is traveling 10 mph slower than twice Brandon's speed; that would mean that 62 is equal to 2 times 36 minus 10. Is that true? Well, 2 times 36 is 72, minus 10 is 62. The answer checks out.

In algebra, there's almost always more than one method of solving a problem. If time allows, it's always a good idea to try to solve the problem using two different methods just to confirm that you've got the answer right.

Speed of Sound

The speed of sound in dry air, v, is given by the equation $v = 331 + 0.6T$, where T is the temperature in Celsius and v is the speed of sound in meters per second.

Example D

Tashi hits a drainpipe with a hammer and 250 meters away Minh hears the sound and hits his own drainpipe. Unfortunately, there is a one second delay between him hearing the sound and hitting his own pipe. Tashi accurately measures the time between her hitting the pipe and hearing Mihn's pipe at 2.46 seconds. What is the temperature of the air?

This is a complex problem and we need to be careful in writing our equations. First of all, the distance the sound travels is equal to the speed of sound multiplied by the time, and the speed is given by the equation above. So the distance equals $(331+0.6T)\times$ time, and the time is $2.46-1$ (because for 1 second out of the 2.46 seconds measured, there was no sound actually traveling). We also know that the distance is 250×2 (because the sound traveled from Tashi to Minh and back again), so our equation is $250\times 2=(331+0.6T)(2.46-1)$, which simplifies to $500=1.46(331+0.6T)$.

Distributing gives us $500=483.26+0.876T$, and subtracting 483.26 from both sides gives us $16.74=0.876T$. Then we divide by 0.876 to get $T\approx 19.1$.

The temperature is about 19.1 degrees Celsius.

Guided Practice

A factory manager is packing engine components into wooden crates to be shipped on a small truck. The truck is designed to hold sixteen crates, and will safely carry a 1200 lb cargo. Each crate weighs 12 lbs empty. How much weight should the manager instruct the workers to put in each crate in order to get the shipment weight as close as possible to 1200 lbs?

Solution:

The unknown quantity is the weight to put in each box, so we'll call that x.

Each crate when full will weigh $x+12$ *lbs*, so all 16 crates together will weigh $16(x+12)$ *lbs*.

We also know that all 16 crates together should weigh 1200 lbs, so we can say that $16(x+12)=1200$.

To solve this equation, we can start by dividing both sides by 16: $x+12=\frac{1200}{16}=75$.

Then subtract 12 from both sides: $x=63$.

The manager should tell the workers to put 63 lbs of components in each crate.

Practice

For 1-6, solve for the variable in the equation.

1. $\frac{s-4}{11}=\frac{2}{5}$
2. $\frac{2k}{7}=\frac{3}{8}$
3. $\frac{7x+4}{3}=\frac{9}{2}$
4. $\frac{9y-3}{6}=\frac{5}{2}$
5. $\frac{r}{3}+\frac{r}{2}=7$
6. $\frac{p}{16}-\frac{2p}{3}=\frac{1}{9}$
7. An engineer is building a suspended platform to raise bags of cement. The platform has a mass of 200 kg, and each bag of cement is 40 kg. He is using two steel cables, each capable of holding 250 kg. Write an equation for the number of bags he can put on the platform at once, and solve it.
8. A scientist is testing a number of identical components of unknown resistance which he labels $x\Omega$. He connects a circuit with resistance $(3x+4)\Omega$ to a steady 12 volt supply and finds that this produces a current of 1.2 amps. What is the value of the unknown resistance?
9. Lydia inherited a sum of money. She split it into five equal parts. She invested three parts of the money in a high-interest bank account which added 10% to the value. She placed the rest of her inheritance plus $500 in the stock market but lost 20% on that money. If the two accounts end up with exactly the same amount of money in them, how much did she inherit?
10. Pang drove to his mother's house to drop off her new TV. He drove at 50 miles per hour there and back, and spent 10 minutes dropping off the TV. The entire journey took him 94 minutes. How far away does his mother live?

CHAPTER 3 Chapter 3: Graphing

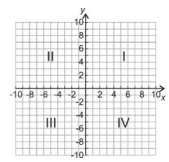
Chapter Outline

3.1 The Coordinate Plane

Learning Objectives

- Identify coordinates of points.
- Plot points in a coordinate plane.
- Graph a function given a table.
- Graph a function given a rule.

Introduction



We now make our transition from a number line that stretches in only one dimension (left to right) to something that exists in two dimensions. The **coordinate plane** can be thought of as two number lines that meet at right angles. The horizontal line is called the $x-$**axis** and the vertical line is the $y-$**axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**. The first quadrant (I) contains all the positive numbers from both axes. It is the top right quadrant. The other quadrants are numbered sequentially (II, III, IV) moving counterclockwise from the first.

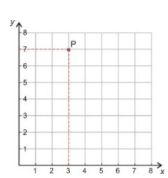
Identify Coordinates of Points

When given a point on a coordinate plane, it is a relatively easy task to determine its **coordinates**. The coordinates of a point are two numbers written together they are called an **ordered pair**. The numbers describe how far along the $x-$axis and $y-$axis the point is. The ordered pair is written in parenthesis, with the $x-$**coordinate** (also called the **ordinate**) first and the $y-$**coordinate** (or the **ordinate**) second.

$(1,7)$	An ordered pair with an x-value of one and a y-value of seven
$(0,5)$	an ordered pair with an x-value of zero and a y-value of five
$(-2.5,4)$	An ordered pair with an x-value of -2.5 and a y-value of four
$(-107.2,-.005)$	An ordered pair with an x-value of -107.2 and a y-value of -0.005.

The first thing to do is realize that identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example 1



Find the coordinates of the point labeled P in the diagram to the right.

Imagine you are standing at the origin (the points where the $x-$axis meets the $y-$axis). In order to move to a position where P was directly above you, you would move 3 units to the **right** (we say this is in the **positive** x direction).

The $x-$coordinate of P is $+3$.

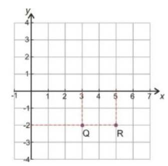
Now if you were standing at the three marker on the $x-$axis, point P would be 7 units above you (above the axis means it is in the **positive** y direction).

The $y-$coordinate of P is $+7$.

Solution

The coordinates of point P are $(3,7)$.

Example 2



Find the coordinates of the points labeled Q and R in the diagram to the right.

In order to get to Q we move three units to the right, in the positive$x$ direction, then two units **down**. This time we are moving in the **negative** y direction. The x coordinate of Q is $+3$, the y coordinate of Q is -2.

The coordinates of R are found in a similar way. The x coordinate is $+5$ (five units in positive x) and the $y-$coordinate is again -2.

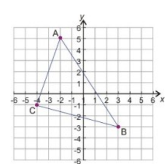
Solution

$Q(3,-2)$

$R(5,-2)$

Example 3

Triangle ABC is shown in the diagram to the right. Find the coordinates of the vertices A, B and C.



Point A:

$x-$coordinate $= -2$

$y-$coordinate $= +5$

Point B:

$x-$coordinate $= +3$

$y-$coordinate $= -3$

Point C:

$x-$coordinate $= -4$

$y-$coordinate $= -1$

Solution

$$A(-2,5)$$
$$B(3,-3)$$
$$C(-4,-1)$$

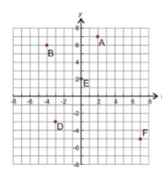
Plot Points in a Coordinate Plane

Plotting points is a simple matter once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example 4

Plot the following points on the coordinate plane.

$$A(2,7) \qquad B(-5,6) \qquad C(-6,0) \qquad D(-3,-3) \qquad E(0,2) \qquad F(7,-5)$$



Point $A(2,7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-5,6)$ is 5 units left, 6 units up. It is in Quadrant II.

Point $C(-6,0)$ is 6 units left, 0 units up. It is **on the** x **axis**.

Point $D(-3,-3)$ is 3 units left, 3 units down. It is in Quadrant III.

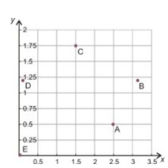
Point $E(0,2)$ is 2 units up from the origin. It is **on the** y **axis**.

Point $F(7,-5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Example 5

Plot the following points on the coordinate plane.

$$A(2.5,0.5) \qquad B(\pi,1.2) \qquad C(2,1.75) \qquad D(0.1,1.2) \qquad E(0,0)$$

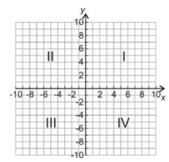


Choice of axes is always important. In Example Four, it was important to have all four quadrants visible. In this case, all the coordinates are positive. There is no need to show the negative values of x or y. Also, there are no x values bigger than about 3.14, and 1.75 is the largest value of y. We will therefore show these points on the following scale $0 \leq x \leq 3.5$ and $0 \leq y \leq 2$. The points are plotted to the right.

Here are some important points to note about this graph.

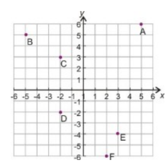
- The tick marks on the axes do not correspond to unit increments (i.e. the numbers do not go up by one).
- The scale on the $x-$axis is different than the scale on the $y-$axis.
- The scale is **chosen** to maximize the clarity of the plotted points.

Lesson Summary



- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the $x-$**axis**) and a vertical number line (the $y-$**axis**). The **origin** is the point where these two lines meet. Four areas, or **quadrants,** are formed as shown in the diagram at right.
- Each point on the coordinate plane has a set of **coordinates,** two numbers written as an **ordered pair** which describe how far along the $x-$axis and $y-$axis the point is. The $x-$**coordinate** is always written first, then the $y-$**coordinate.** Here is an exaxmple (x,y).

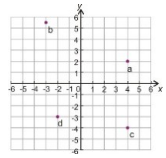
Review Questions



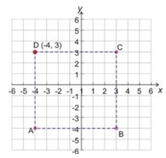
1. Identify the coordinates of each point, $A-F$, on the graph to the right.
2. Plot the following points on a graph and identify which quadrant each point lies in:
   (a) $(4,2)$
   (b) $(-3,5.5)$
   (c) $(4,-4)$
   (d) $(-2,-3)$
3. The following three points are three vertices of square $ABCD$. Plot them on a graph then determine what the coordinates of the fourth point, D, would be. Plot that point and label it.
   $A\ (-4,-4)$
   $B\ (3,-4)$
   $C\ (3,3)$
4. Becky has a large bag of MMs that she knows she should share with Jaeyun. Jaeyun has a packet of Starburst. Becky tells Jaeyun that for every Starburst he gives her, she will give him three MMs in return. If x is the number of Starburst that Jaeyun gives Becky, and $y$is the number of MMs he gets in return then complete each of the following.
   (a) Write an algebraic rule for y in terms of x
   (b) Make a table of values for y with x values of $0,1,2,3,4,5$.
   (c) Plot the function linking x *and* y on the following scale $0 \le x \le 10, 0 \le y \le 10$.

Review Answers

1. $A(5,6)B(-5,5)C(-2,3)D(-2,-2)E(3,-4)F(2,-6)$

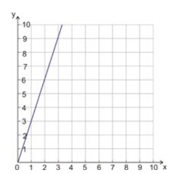
2. 

(a) Quadrant I
(b) Quadrant II
(c) Quadrant IV
(d) Quadrant III

3. 

4. (a) $y = 3x$
(b)

x	y
0	0
1	3
2	6
3	9
4	12
5	15

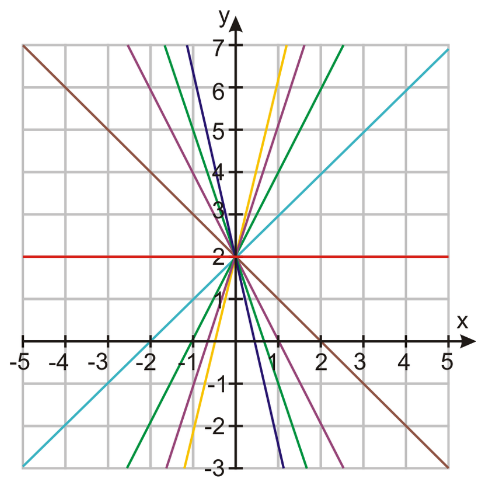
(c) 

3.2 Graphing Using Intercepts

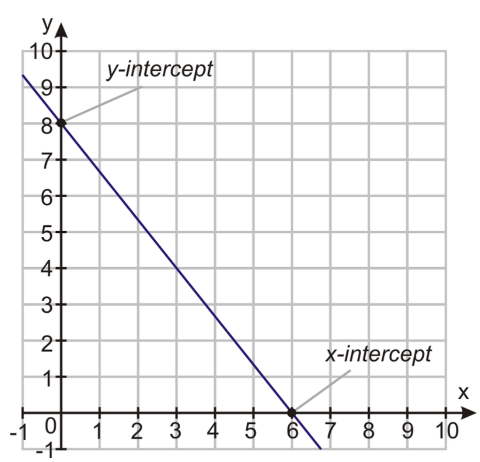
Learning Objectives

- Find intercepts of the graph of an equation.
- Use intercepts to graph an equation.
- Solve real-world problems using intercepts of a graph.

Introduction



Only two distinct points are needed to uniquely define a graph of a line. After all, there are an infinite number of lines that pass through a single point (a few are shown in the graph above). But if you supplied just one more point, there can only be one line that passes through both points. To plot the line, just plot the two points and use a ruler, edge placed on both points, to trace the graph of the line.

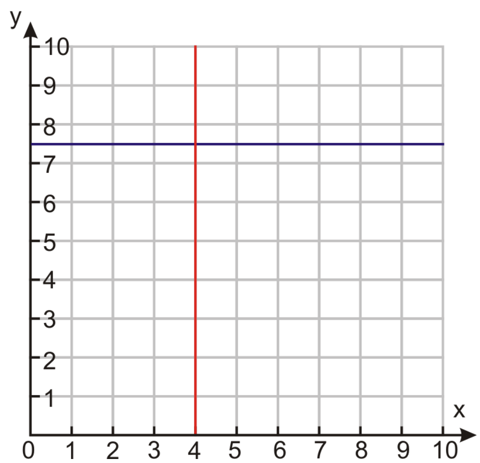


There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, we will focus on two points that are rather convenient for graphing: the points where our line crosses the x and y axes, or **intercepts.** We will be finding intercepts algebraically and using them to quickly plot graphs.

Look at the graph above. The $y-$**intercept** occurs at the point where the graph crosses the $y-$axis. The $y-$value at this point is 8.

Similarly the $x-$**intercept** occurs at the point where the graph crosses the $x-$axis. The $x-$value at this point is 6.

Now we know that the x value of all the points on the $y-$axis is zero, and the y value of all the points on the $x-$axis is also zero. So if we were given the coordinates of the two intercepts (0, 8) and (6, 0) we could quickly plot these points and join them with a line to recreate our graph.



Note: Not all lines will have both intercepts but most do. Specifically, horizontal lines never cross the $x-$axis and vertical lines never cross the $y-$axis. For examples of these special case lines, see the graph above.

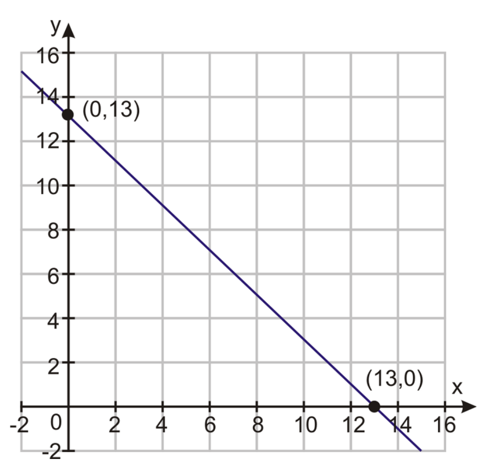
Finding Intercepts

Example 1

Find the intercepts of the line $y = 13 - x$ *and use them to graph the function.*

The first intercept is easy to find. The $y-$intercept occurs when $x = 0$ Substituting gives:

$$y = 13 - 0 = 13 \qquad (0, 13) \text{ is the } x - \text{intercept.}$$



We know that the $x-$intercept has, by definition, a $y-$value of zero. Finding the corresponding $x-$value is a simple case of substitution:

$$\begin{aligned} 0 &= 13 - x && \text{To isolate } x \text{ subtract 13 from both sides.} \\ -13 &= -x && \text{Divide by } -1. \end{aligned}$$

Solution

(13, 0) is the $x-$intercept.

To draw the graph simply plot these points and join them with a line.

Example 2

Graph the following functions by finding intercepts.

a. $y = 2x + 3$

b. $y = 7 - 2x$

c. $4x - 2y = 8$

d. $2x + 3y = -6$

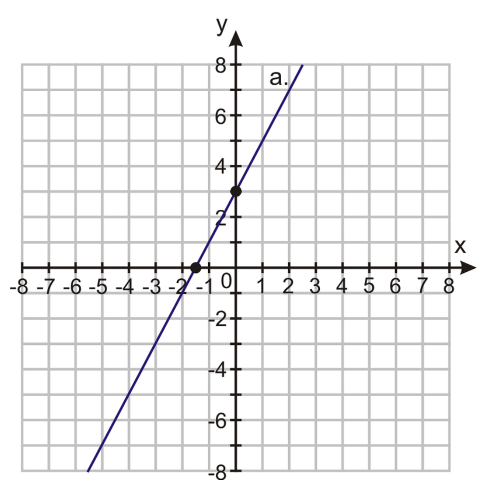
Solutions

a. Find the $y-$intercept by plugging in $x = 0$.

$$y = 2 \cdot 0 + 3 = 3 \qquad \text{The } y-\text{intercept is } (0, 3)$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 0 &= 2x + 3 && \text{Subtract 3 from both sides.} \\ -3 &= 2x && \text{Divide by 2.} \\ -\frac{3}{2} &= x && \text{The } x-\text{intercept is } (-1.5, 0). \end{aligned}$$

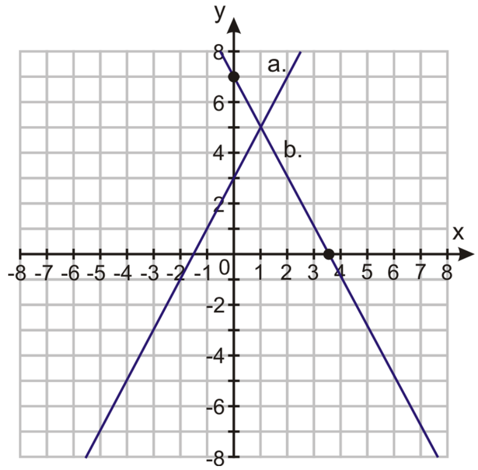


b. Find the $y-$intercept by plugging in $x = 0$.

$$y = 7 - 2 \cdot 0 = 7 \qquad \text{The } y-\text{intercept is } (0, 7).$$

Find the $x-$intercept by plugging in $y=0$.

$$\begin{aligned} 0 &= 7-2x && \text{Subtract 7 from both sides.}\\ -7 &= -2x && \text{Divide by } -2.\\ \frac{7}{2} &= x && \text{The } x-\text{intercept is } (3.5,0). \end{aligned}$$

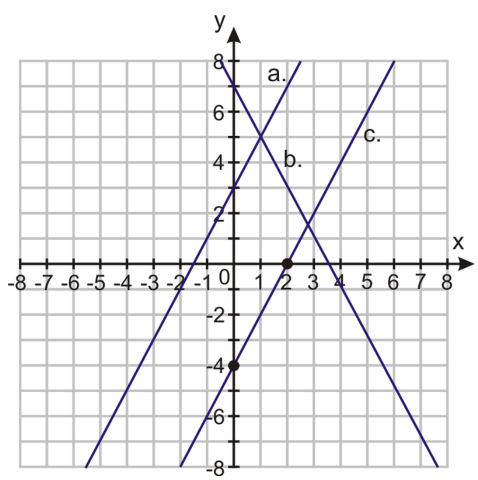


c. Find the $y-$intercept by plugging in $x=0$.

$$\begin{aligned} 4\cdot 0-2y &= 8 && \\ -2y &= 8 && \text{Divide by } -2.\\ y &= -4 && \text{The } y-\text{intercept is } (0,-4). \end{aligned}$$

Find the $x-$intercept by plugging in $y=0$.

$$\begin{aligned} 4x-2\cdot 0 &= 8 && \\ 4x &= 8 && \text{Divide by } 4.\\ x &= 2 && \text{The } x-\text{intercept is } (2,0). \end{aligned}$$

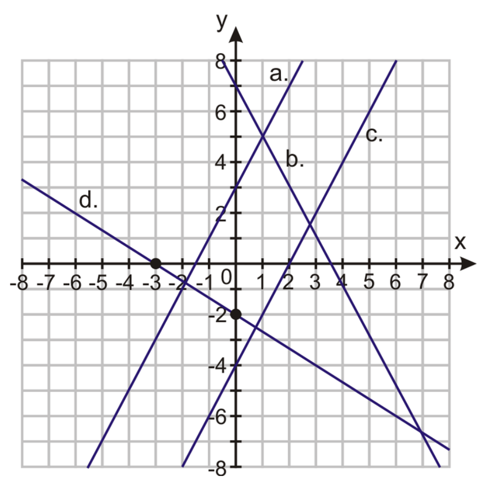


d. Find the $y-$intercept by plugging in $x = 0$.

$$\begin{aligned} 2 \cdot 0 + 3y &= -6 \\ 3y &= -6 && \text{Divide by 3.} \\ y &= -2 && \text{The } y-\text{intercept is } (0, -2). \end{aligned}$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 2x + 3 \cdot 0 &= -6 \\ 2x &= -6 && \text{Divide by 2.} \\ x &= -3 && \text{The } x-\text{intercept is } (-3, 0) \end{aligned}$$



Solving Real-World Problems Using Intercepts of a Graph

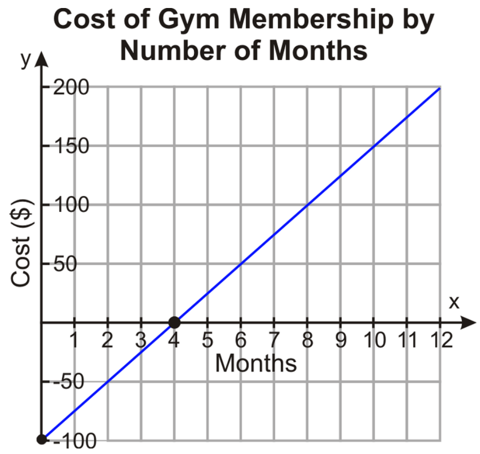
Example 3

The monthly membership cost of a gym is \$25 per month. To attract members, the gym is offering a \$100 cash rebate if members sign up for a full year. Plot the cost of gym membership over a 12 month period. Use the graph to determine the final cost for a 12 month membership.

Let us examine the problem. Clearly the cost is a function of the number of months (not the other way around). Our independent variable is the number of months (the domain will be whole numbers) and this will be our x value. The cost in dollars is the dependent variable and will be our y value. Every month that passes the money paid to the gym goes up by \$25. However, we start with a \$100 cash gift, so our **initial cost** ($y-$intercept) is \$100. This pays for four months $(4 \times \$25 = 100)$ so after four months the cost of membership ($y-$value) is zero.

The $y-$intercept is (0, -100). The $x-$intercept is (4, 0).

We plot our points, join them with a straight line and extend that line out all the way to the $x = 12$ line. The graph is shown below.



To find the cost of a 12 month membership we simply read off the value of the function at the 12 month point. A line drawn up from $x = 12$ on the x axis meets the function at a y value of \$200.

Solution

The cost of joining the gym for one year is \$200.

Example 4

Jesus has \$30 to spend on food for a class barbeque. Hot dogs cost \$0.75 each (including the bun) and burgers cost \$1.25 (including bun and salad). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than \$30.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that John buys is x, then the money spent on burgers is $1.25x$.

If the number of hot dogs he buys is y then the money spent on hot dogs is $0.75y$.

$$1.25x + 0.75y \qquad \text{The total cost of the food.}$$

The total amount of money he has to spend is \$30. If he is to spend it ALL, then we can use the following equation.

$$1.25x + 0.75y = 30$$

We solve for the intercepts using the cover-up method.

First the $y-$intercept $(x = 0)$.

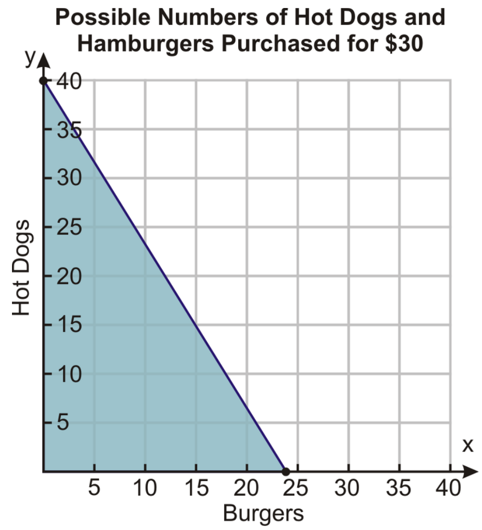
$1.25 \cdot 0 + 0.75y = 30$

$0.75y = 30$
$y = 40 \qquad y - \text{intercept } (0,40)$

Then the $x-$intercept $(y = 0)$

$1.25x + 0.75 \cdot 0 = 30$

$1.25x = 30$
$x = 24 \qquad x - \text{intercept } (24,0)$



We can now plot the points and join them to create our graph, shown right.

Here is an alternative to the equation method.

If Jesus were to spend ALL the money on hot dogs, he could buy $\frac{30}{0.75} = 40$ hot dogs. If on the other hand, he were to buy only burgers, he could buy $\frac{30}{1.25} = 24$ burgers. So you can see that we get two intercepts: (0 burgers, 40 hot dogs) and (24 burgers, 0 hot dogs). We would plot these in an identical manner and design our graph that way.

As a final note, we should realize that Jesus' problem is really an example of an **inequality.** He can, in fact, spend any amount up to $30. The only thing he cannot do is spend more than $30. So our graph reflects this. The shaded region shows where Jesus' solutions all lie. We will see more inequalities in a later section.

Lesson Summary

- A $y-$**intercept** occurs at the point where a graph crosses the $y-$axis $(x = 0)$ and an $x-$**intercept** occurs at the point where a graph crosses the $x-$axis $(y = 0)$.
- The $y-$intercept can be found by **substituting** $x = 0$ into the equation and solving for y. Likewise, the $x-$intercept can be found by **substituting** $y = 0$ into the equation and solving for x.
  - Note: A linear equation is in **standard form** if it is written as "positive coefficient times x plus (or minus) positive coefficient times y equals value". $(Ax + By = C)$

Review Questions

Find the intercepts for the following equations

1. $y = 3x - 6$
2. $y = -2x + 4$
3. $y = 14x - 21$
4. $y = 7 - 3x$
5. $5x - 6y = 15$
6. $3x - 4y = -5$
7. $2x + 7y = -11$
8. $5x + 10y = 25$

Find the intercepts and then graph the following equations.

9. $y = 2x + 3$

10. $6(x-1) = 2(y+3)$
11. $x-y = 5$
12. $x+y = 8$
13. At the local grocery store strawberries cost \$3.00 per pound and bananas cost \$1.00 per pound. If I have \$10 to spend between strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly \$10.
14. A movie theater charges \$7.50 for adult tickets and \$4.50 for children. If the theater takes \$900 in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
15. Why can't we use the intercept method to graph the following equation? $3(x+2) = 2(y+3)$

3.3 Slope

Here you'll learn how to find the slope of a line given the line's graph or two of its points.

What if you were given two points that a line passes through like (-1, 0) and (2, 2)? How could you find the slope of that line? After completing this Concept, you'll be able to find the slope of any line.

Guidance

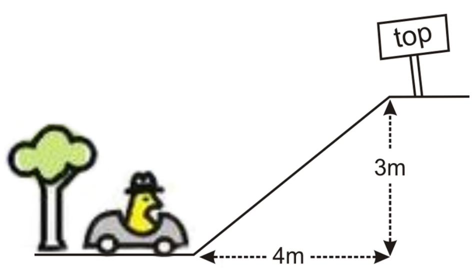
Wheelchair ramps at building entrances must have a slope between $\frac{1}{16}$ and $\frac{1}{20}$. If the entrance to a new office building is 28 inches off the ground, how long does the wheelchair ramp need to be?

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, or the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.

$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

To make it easier to remember, we often word it like this:

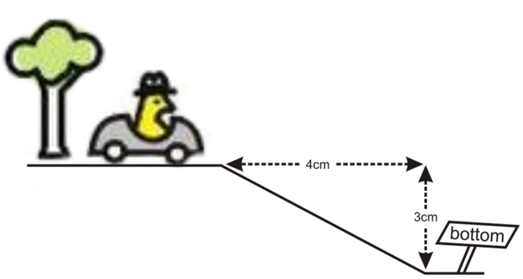
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$



In the picture above, the slope would be the ratio of the **height** of the hill to the horizontal **length** of the hill. In other words, it would be $\frac{3}{4}$, or 0.75.

If the car were driving to the **right** it would **climb** the hill - we say this is a positive slope. Any time you see the graph of a line that goes up as you move to the right, the slope is **positive**.

If the car kept driving after it reached the top of the hill, it might go down the other side. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**.



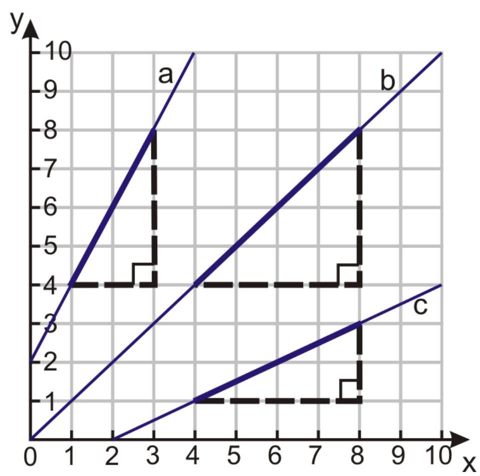
Here's where it gets tricky: If the car turned around instead and drove back down the left side of the hill, the slope of that side would still be positive. This is because the rise would be -3, but the run would be -4 (think of the $x-$axis - if you move from right to left you are moving in the **negative** $x-$direction). That means our slope ratio would be $\frac{-3}{-4}$, and the negatives cancel out to leave 0.75, the same slope as before. In other words, the slope of a line is the same no matter which direction you travel along it.

Find the Slope of a Line

A simple way to find a value for the slope of a line is to draw a right triangle whose hypotenuse runs along the line. Then we just need to measure the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example A

Find the slopes for the three graphs shown.



Solution

There are already right triangles drawn for each of the lines - in future problems you'll do this part yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. points whose coordinates are all integers).

a) The rise shown in this triangle is 4 units; the run is 2 units. The slope is $\frac{4}{2} = 2$.

b) The rise shown in this triangle is 4 units, and the run is also 4 units. The slope is $\frac{4}{4} = 1$.

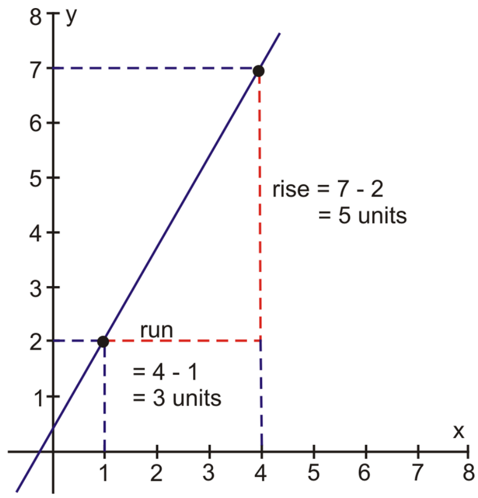
c) The rise shown in this triangle is 2 units, and the run is 4 units. The slope is $\frac{2}{4} = \frac{1}{2}$.

Example B

Find the slope of the line that passes through the points (1, 2) and (4, 7).

Solution

We already know how to graph a line if we're given two points: we simply plot the points and connect them with a line. Here's the graph:



Since we already have coordinates for the vertices of our right triangle, we can quickly work out that the rise is $7-2=5$ and the run is $4-1=3$ (see diagram). So the slope is $\frac{7-2}{4-1}=\frac{5}{3}$.

If you look again at the calculations for the slope, you'll notice that the 7 and 2 are the $y-$coordinates of the two points and the 4 and 1 are the $x-$coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1,y_1) and (x_2,y_2):

Slope between (x_1,y_1) and $(x_2,y_2) = \frac{y_2-y_1}{x_2-x_1}$

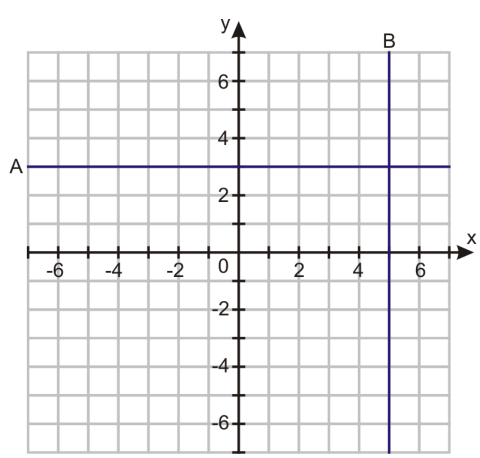
or $m=\frac{\Delta y}{\Delta x}$.

In the second equation the letter m denotes the slope (this is a mathematical convention you'll see often) and the Greek letter delta (Δ) means ***change***. So another way to express slope is *change in y* divided by *change in x*. In the next section, you'll see that it doesn't matter which point you choose as point 1 and which you choose as point 2.

Find the Slopes of Horizontal and Vertical lines

Example C

Determine the slopes of the two lines on the graph below.



Solution

There are 2 lines on the graph: $A(y=3)$ and $B(x=5)$.

Let's pick 2 points on line A—say, $(x_1,y_1)=(-4,3)$ and $(x_2,y_2)=(5,3)$—and use our equation for slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (3)}{(5) - (-4)} = \frac{0}{9} = 0.$$

If you think about it, this makes sense - if y doesn't change as x increases then there is no slope, or rather, the slope is zero. You can see that this must be true for all horizontal lines.

Horizontal lines (y = *constant*) all have a slope of 0.

Now let's consider line B. If we pick the points $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (5, 4)$, our slope equation is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4)-(-3)}{(5)-(5)} = \frac{7}{0}$. But dividing by zero isn't allowed!

In math we often say that a term which involves division by zero is **undefined.** (Technically, the answer can also be said to be infinitely large—or infinitely small, depending on the problem.)

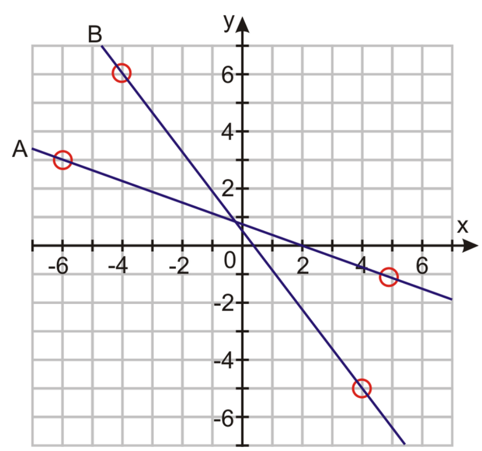
Vertical lines ($x = constant$) all have an infinite (or undefined) slope.

Vocabulary

- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as "m".
- Slope can be expressed as $\frac{\text{rise}}{\text{run}}$, or $\frac{\Delta y}{\Delta x}$.
- The slope between two points (x_1, y_1) and (x_2, y_2) is equal to $\frac{y_2 - y_1}{x_2 - x_1}$.
- **Horizontal lines** (where $y = a$ constant) all have a slope of 0.
- **Vertical lines** (where $x = a$ constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity.**

Guided Practice

Find the slopes of the lines on the graph below.



Solution

Look at the lines - they both slant down (or decrease) as we move from left to right. Both these lines have **negative slope.**

The lines don't pass through very many convenient lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been circled on the graph, and we'll use them to determine the slope. We'll also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For Line A:

$$(x_1, y_1) = (-6, 3) \qquad (x_2, y_2) = (5, -1) \qquad\qquad (x_1, y_1) = (5, -1) \qquad (x_2, y_2) = (-6, 3)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (3)}{(5) - (-6)} = \frac{-4}{11} \approx -0.364 \qquad\qquad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (-1)}{(-6) - (5)} = \frac{4}{-11} \approx -0.364$$

For Line B

$$(x_1, y_1) = (-4, 6) \qquad (x_2, y_2) = (4, -5) \qquad\qquad (x_1, y_1) = (4, -5) \qquad (x_2, y_2) = (-4, 6)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-5) - (6)}{(4) - (-4)} = \frac{-11}{8} = -1.375 \qquad\qquad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(6) - (-5)}{(-4) - (4)} = \frac{11}{-8} = -1.375$$

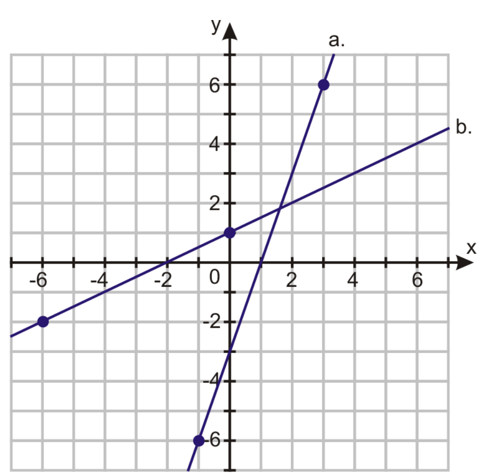
You can see that whichever way round you pick the points, the answers are the same. Either way, **Line *A* has slope -0.364, and Line *B* has slope -1.375.**

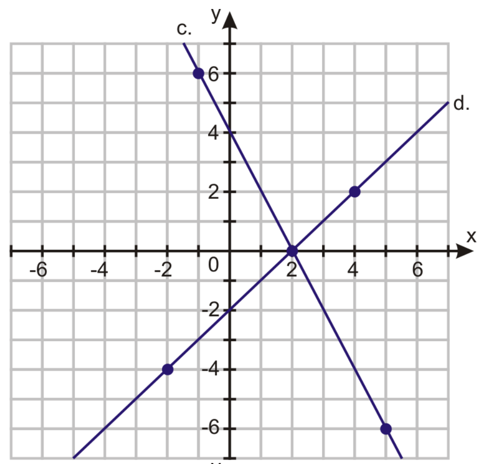
Practice

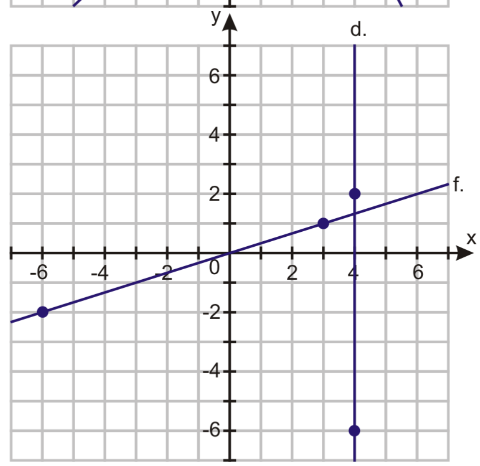
Use the slope formula to find the slope of the line that passes through each pair of points.

1. (-5, 7) and (0, 0)
2. (-3, -5) and (3, 11)
3. (3, -5) and (-2, 9)
4. (-5, 7) and (-5, 11)
5. (9, 9) and (-9, -9)
6. (3, 5) and (-2, 7)
7. (2.5, 3) and (8, 3.5)

For each line in the graphs below, use the points indicated to determine the slope.

8. 

9. 

10. 

11. For each line in the graphs above, imagine another line with the same slope that passes through the point (1, 1), and name one more point on that line.

3.4 Using Slope-Intercept Form

Here you'll use slope-intercept form and identify the slope and the y-intercept.

Guidance

We have seen linear equations in function form, have created tables of values and graphs to represent them, looked at their x- and y-intercepts, and studied their slopes. One of the most useful forms of a linear equation is the ***slope-intercept form*** which we will be using with standard form in this Concept.

Remember ***standard form***?

The standard form of an equation is when the equation is written in $Ax+By=C$ form.

This form of the equation allows us to find many possible solutions. In essence, we could substitute any number of values for x and y and create the value for C. When an equation is written in standard form, it is challenging for us to determine the slope and the y –intercept.

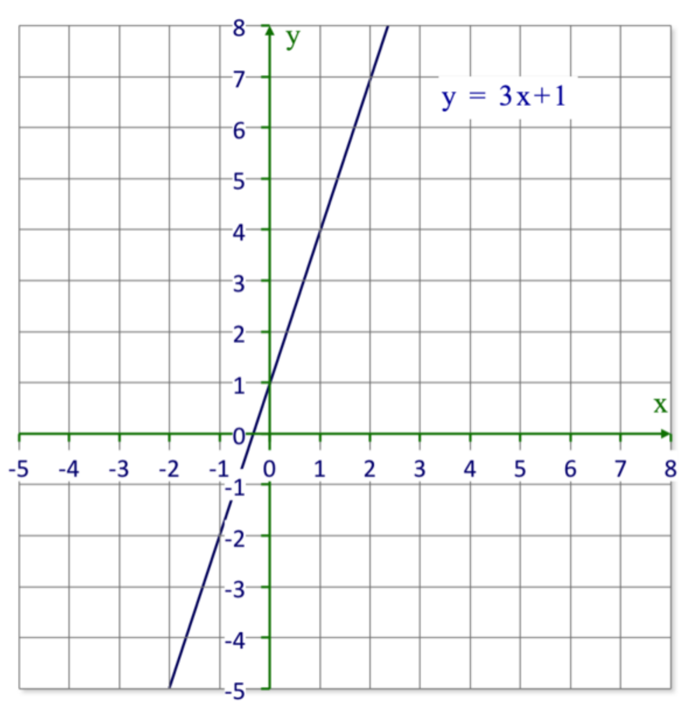
Think back, remember that the *slope* is the steepness of the line and the y *–intercept* is the point where the line crosses the y –axis.

We can write an equation in a different form than in standard form. This is when $y =$ an equation. We call this form of an equation *slope –intercept form* .

Slope –Intercept Form is $y = mx+b$ –where m is the slope and b is the y –intercept.

Take a look at this graph and equation.

Graph the line $y = 3x+1$



Here we can calculate the slope of the line using the rise over the run and see that it is 3. The y –intercept is 1. Notice that we can find these values in our equation too.

When an equation is in slope –intercept form, we can spot the slope and the y –intercept by looking at the equation.

$y = mx + b$

Here m is the value of the slope and b is the value of the y –intercept.

For any equation written in the form $y = mx + b$, m is the slope and b is the y-intercept. For that reason, $y = mx + b$ is called the *slope-intercept form.* Using the properties of equations, you can write any equation in this form.

Because we can use slope –intercept form, we can rewrite equations in standard form into slope –intercept form. Then we can easily determine the slope and y –intercept of each equation.

Take a look here.

Write $4x + 2y = 6$ in slope –intercept form. Then determine the slope and the y –intercept by using the equation.

$$\begin{aligned}
4x + 2y &= 6 \\
4x + 2y - 2y &= 6 - 2y \\
4x &= 6 - 2y \\
4x - 6 &= -2y \\
\frac{4x - 6}{-2} &= y \\
y &= -2x + 3
\end{aligned}$$

Now we can determine the slope and the y –intercept from the equation.

$$\begin{aligned}
-2 &= \text{slope} \\
3 &= y - \text{intercept}
\end{aligned}$$

Think back to our work with functions. Remember how we could write a function in function form? Well take a look at function form compared with slope –intercept form.

Function form $= f(x) = 2x + 1$

Slope –Intercept Form $= y = 2x + 1$

Yes! The two are the same. These two equations are equivalent!

Determine the slope and the y-intercept in each equation.

Example A

$y = x + 4$

Solution: slope = 1, y-intercept = 4

Example B

$2x + y = 10$

Solution: slope = -2, y-intercept = 10

Example C

$-3x+y=9$

Solution: slope = 3, y-intercept = 9

Now let's go back to the dilemma at the beginning of the Concept.

$y=-2x-8$

Looking at this equation, you can see that the slope is -2 and the y-intercept is 8.

Vocabulary

Slope –Intercept Form
the form of an equation $y=mx+b$

Standard Form
the form of an equation $Ax+By=C$

Slope
the steepness of the line, calculated by the ratio of rise over run.

y –Intercept
the point where a line crosses the $y$axis.

Guided Practice

Here is one for you to try on your own.

Write this equation in slope-intercept form and then determine the slope and the y-intercept.

$$\begin{aligned}
-18x+6y &= 12\\
-18x+6y &= 12\\
-18x+6y+18x &= 18x+12\\
6y &= 18x+12\\
6y &= 18x+12\\
\frac{18x+12}{6} &= y\\
y &= 3x+2
\end{aligned}$$

Given this equation, the slope is 3 and the y-intercept is 2.

Practice

Directions: Look at each equation and identify the slope and the y –intercept by looking at each equation. There are two answers for each problem.

1. $y = 2x + 4$
2. $y = 3x - 2$
3. $y = 4x + 3$
4. $y = 5x - 1$
5. $y = \frac{1}{2}x + 2$
6. $y = -2x + 4$
7. $y = -3x - 1$
8. $y = \frac{-1}{3}x + 5$

Directions: Use what you have learned to write each in slope –intercept form and then answer each question.

9. $2x + 4y = 12$
10. Write this equation in slope –intercept form.
11. What is the slope?
12. What is the y –intercept?
13. $6x + 3y = 24$
14. Write this equation in slope –intercept form.
15. What is the slope?
16. What is the y –intercept?
17. $5x + 5y = 15$
18. Write this equation in slope –intercept form.
19. What is the slope?
20. What is the y –intercept?

3.5 Direct Variation Models

Learning Objectives

- Identify direct variation.
- Graph direct variation equations.
- Solve real-world problems using direct variation models.

Introduction

Suppose you see someone buy five pounds of strawberries at the grocery store. The clerk weighs the strawberries and charges \$12.50 for them. Now suppose you wanted two pounds of strawberries for yourself. How much would you expect to pay for them?

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a "per pound" basis, and that if you buy two-fifths of the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries.

$$\frac{2}{5} \times \$12.50 = \$5.00$$

Similarly, if you bought 10 pounds of strawberries (twice the amount) you would pay $2 \times \$12.50$ and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x, then we write the relationship as:

$$y = k \cdot x$$

k is called the **constant of proportionality**.

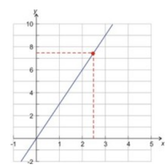
If we were to graph this function you can see that it passes through the origin, because $y = 0$, when $x = 0$ whatever the value of k. So we know that a direct variation, when graphed, has a single intercept at $(0, 0)$.

Example 1

If y varies directly with x according to the relationship $y = k \cdot x$, and $y = 7.5$ when $x = 2.5$, determine the constant of proportionality, k.

We can solve for the constant of proportionality using substitution.

Substitute $x = 2.5$ and $y = 7.5$ into the equation $y = k \cdot x$



$$\begin{aligned} 7.5 &= k(2.5) && \text{Divide both sides by 2.5.} \\ \frac{7.5}{2.5} &= k = 3 \end{aligned}$$

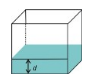
Solution

The constant of proportionality, $k = 3$.

We can graph the relationship quickly, using the intercept $(0,0)$ and the point $2.5, 7.5)$. The graph is shown right. It is a straight line with a slope $= 3$.

The graph of a direct variation has a slope that is equal to the constant of proportionality, k.

Example 2



The volume of water in a fish-tank, V, varies directly with depth, d. If there are 15 *gallons in the tank when the depth is eight inches, calculate how much water is in the tank when the depth is* 20 *inches.*

This is a good example of a direct variation, but for this problem we will need to determine the equation of the variation ourselves. Since the volume, V, depends on depth, d, we will use the previous equation to create new one that is better suited to the content of the new problem.

$$\begin{aligned} y &= k \cdot x && \text{In place of } y \text{ we will use } V \text{ and in place of } x \text{ we will use d.} \\ V &= k \cdot d \end{aligned}$$

We know that when the depth is 8 inches, the volume is 15 gallons. Now we can substitute those values into our equation.

Substitute $V = 15$ and $x = 8$:

$$\begin{aligned} V &= k \cdot d \\ 15 &= k(8) && \text{Divide both sides by8.} \\ \frac{15}{8} &= k = 1.875 \end{aligned}$$

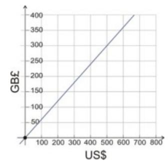
Now to find the volume of water at the final depth we use $V = k \cdot d$ and substitute for our new d.

$$\begin{aligned} V &= k \cdot d \\ V &= 1.875 \times 20 \\ V &= 37.5 \end{aligned}$$

Solution

At a depth of 20 inches, the volume of water in the tank is 37.5 gallons.

Example 3



The graph shown to the right shows a conversion chart used to convert U.S. dollars (US\$) to British pounds (GB) in a bank on a particular day. Use the chart to determine the following.

(i) The number of pounds you could buy for \$600.

(ii) The number of dollars it would cost to buy 200.

(iii) The exchange rate in pounds per dollar.

(iv) Is the function continuous or discrete?

Solution

In order to solve (i) and (ii) we could simply read off the graph: it looks as if at $x = 600$ the graph is about one fifth of the way between 350 and 400. So \$600 would buy 360. Similarly, the line $y = 200$ would appear to intersect the graph about a third of the way between \$300 and \$400. We would probably round this to \$330. So it would cost approximately \$330 to buy 200.

To solve for the exchange rate we should note that as this is a direct variation, because the graph is a straight line passing through the origin. The slope of the line gives the constant of proportionality (in this case the **exchange rate**) and it is equal to the ratio of the $y-$value to $x-$value. Looking closely at the graph, it is clear that there is one lattice point that the line passes through $(500, 300)$. This will give us the most accurate estimate for the slope (exchange rate).

$$y = k \cdot x \Rightarrow k = \frac{y}{x}$$
$$\text{rate} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar}$$

Graph Direct Variation Equations

We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of proportionality, k. Graphing is a simple matter of using the point-slope or point-point methods discussed earlier in this chapter.

Example 4

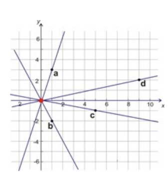
Plot the following direct relations on the same graph.

a. $y = 3x$

b. $y = -2x$

c. $y = -0.2x$

d. $y = \frac{2}{9}x$



Solution

a. The line passes through $(0,0)$. All these functions will pass through this point. It is plotted in red. This function has a slope of 3. When we move across by one unit, the function increases by three units.

b. The line has a slope of -2. When we move across the graph by one unit the function **falls** by two units.

c. The line has a slope of -0.2. As a fraction this is equal to $-\frac{1}{5}$ When we move across by five units, the function **falls** by one unit.

d. The line passes through $(0,0)$ and has a slope of $\frac{2}{9}$. When we move across the graph by 9 units, the function increases by two units.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time that we have one quantity that doubles when another related quantity doubles, we say that they follow a direct variation.

Newton's Second Law

In 1687, Sir Isaac Newton published the famous *Principea Mathematica*. It contained, among other things, his Second Law of Motion. This law is often written as:

$$F = m \cdot a$$

A force of F (Newtons) applied to a mass of m (kilograms) results in acceleration of $a(\text{meterspersecond}^2)$.

Example 5

If a 175 *Newton force causes a heavily loaded shopping cart to accelerate down the aisle with an acceleration of* 2.5 m/s^2, *calculate*

(i) The mass of the shopping cart.

(ii) The force needed to accelerate the same cart at 6 m/s^2 .

Solution

(i) This question is basically asking us to solve for the constant of proportionality. Let us compare the two formulas.

$$\begin{aligned} y &= k \cdot x && \text{The direct variation equation} \\ F &= m \cdot a && \text{Newton's Second law} \end{aligned}$$

We see that the two equations have the same form; y is analogous to force and x analogous to acceleration.

We can solve for m (the mass) by substituting our given values for force and acceleration:

Substitute $F = 175$, $a = 2.5$

$$\begin{aligned} 175 &= m(2.5) && \text{Divide both sides by 2.5.} \\ 70 &= m \end{aligned}$$

The mass of the shopping cart is 70 kg.

(ii) Once we have solved for the mass we simply substitute that value, plus our required acceleration back into the formula $F = m \cdot a$ and solve for F:

Substitute $m = 70$, $a = 6$

$$F = 70 \times 6 = 420$$

The force needed to accelerate the cart at 6 m/s^2 is 420 Newtons.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship:

$$V = I \cdot R \qquad \text{where R is the resistance (measured in Ohms)}$$

The resistance is considered to be a constant for all values of V and I.

Example 6

A certain electronics component was found to pass a current of 1.3 *amps at a voltage of* 2.6 *volts. When the voltage was increased to* 12.0 *volts the current was found to be* 6.0 *amps.*

a) Does the component obey Ohms law?

b) What would the current be at 6 *volts?*

Solution

a) Ohm's law is a simple direct proportionality law. Since the resistance R is constant, it acts as our constant of proportionality. In order to know if the component obeys Ohm's law we need to know if it follows a direct proportionality rule. In other words **is** V **directly proportional to** I?

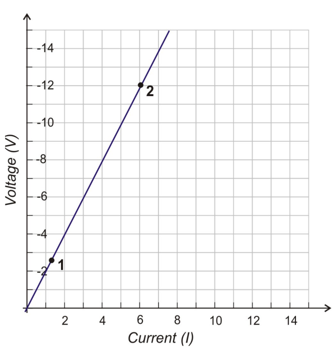


FIGURE 3.1

Method One Graph It

If we plot our two points on a graph and join them with a line, does the line pass through $(0,0)$?

Point 1 $= 2.6$, $I = 1.3$ our point is $(1.3, 2.6)^*$

Point 2 $V = 12.0$, $I = 6.0$ our point is $(6, 12)$

Plotting the points and joining them gives the following graph.

The graph does appear to pass through the origin, so

Yes, the component obeys Ohms law.

Method Two Solve for

We can quickly determine the value of R in each case. It is the ratio of the voltage to the resistance.

Case 1 $R = \frac{V}{I} = \frac{2.6}{1.3} = 2$ Ohms

Case 2 $R = \frac{V}{I} = \frac{12}{6} = 2$ Ohms

The values for R agree! This means that the line that joins point 1 to the origin is the same as the line that joins point 2 to the origin. **The component obeys Ohms law**.

b) To find the current at 6 volts, simply substitute the values for V and R into $V = I \cdot R$

Substitute $V = 6$, $R = 2$

- In physics, it is customary to plot voltage on the horizontal axis as this is most often the independent variable. In that situation, the slope gives the **conductance, σ**. However, by plotting the current on the horizontal axis, the **slope** is equal to the **resistance, R**.

$$\begin{aligned} 6 &= I(2) && \text{Divide both sides by 2.} \\ 3 &= I \end{aligned}$$

Solution

The current through the component at a voltage of 6 volts is 3 amps.

Lesson Summary

- If a variable y varies ***directly*** with variable x, then we write the relationship as

$$y = k \cdot x$$

Where k is a constant called the **constant of proportionality.**

- **Direct variation** is very common in many areas of science.

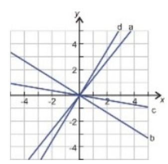
Review Questions

1. Plot the following direct variations on the same graph.
   (a) $y = \frac{4}{3}x$
   (b) $y = -\frac{2}{3}x$
   (c) $y = -\frac{1}{6}x$

(d) $y = 1.75x$

2. Dasans mom takes him to the video arcade for his birthday. In the first 10 minutes, he spends \$3.50 playing games. If his allowance for the day is \$20.00, how long can he keep playing games before his money is gone?
3. The current standard for low-flow showerheads heads is 2.5 gallons per minute. Calculate how long it would take to fill a 30 gallon bathtub using such a showerhead to supply the water.
4. Amen is using a hose to fill his new swimming pool for the first time. He starts the hose at 10 P.M. and leaves it running all night. At 6 AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
5. Land in Wisconsin is for sale to property investors. A 232 acre lot is listed for sale for \$200500. Assuming the same price per acre, how much would a 60 acre lot sell for?
6. The force (F) needed to stretch a spring by a distance x is given by the equation $F = k \cdot x$, where k is the spring constant (measured in Newtons per centimeter, $\mathrm{N/cm}$). If a 12 Newton force stretches a certain spring by 10 cm, calculate:
   (a) The spring constant, k
   (b) The force needed to stretch the spring by 7 cm .
   (c) The distance the spring would stretch with a 23 Newton force.

Review Answers

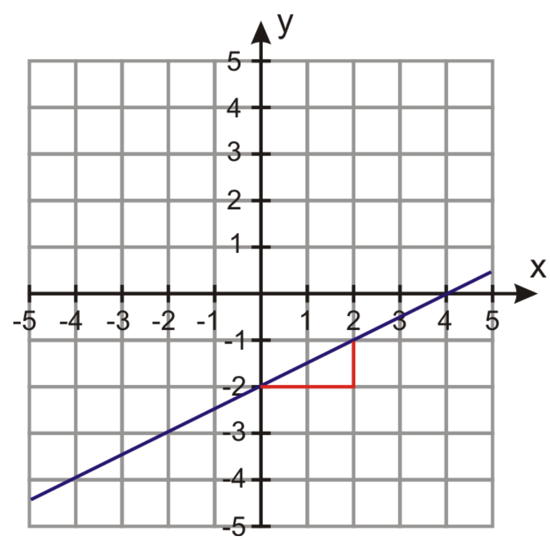
1. 

2. 57 minutes 8 seconds
3. 12 minutes
4. 12 : 00 Midday
5. \$51, 853
6. (a) $k = 1.2\ \mathrm{N/cm}$
   (b) 8.4 Newtons
   (c) 19.17 cm

3.6 Forms of Linear Equations

You've already learned how to write an equation in slope–intercept form: simply start with the general equation for the slope-intercept form of a line, $y = mx + b$, and then plug the given values of $m$and $b$into the equation. For example, a line with a slope of 4 and a $y-$intercept of -3 would have the equation $y = 4x - 3$.

If you are given just the graph of a line, you can read off the slope and $y-$intercept from the graph and write the equation from there. For example, on the graph below you can see that the line rises by 1 unit as it moves 2 units to the right, so its slope is $\frac{1}{2}$. Also, you can see that the $y-$intercept is -2, so the equation of the line is $y = \frac{1}{2}x - 2$.



Write an Equation Given the Slope and a Point

Often, we don't know the value of the $y-$intercept, but we know the value of y for a non-zero value of x. In this case, it's often easier to write an equation of the line in **point-slope form.** An equation in point-slope form is written as $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the line.

Example 1

A line has a slope of $\frac{3}{5}$, and the point (2, 6) is on the line. Write the equation of the line in point-slope form.

Solution

Start with the formula $y - y_0 = m(x - x_0)$.

Plug in $\frac{3}{5}$ for m, 2 for x_0 and 6 for y_0.

The equation in point-slope form is $y - 6 = \frac{3}{5}(x - 2)$.

Notice that the equation in point-slope form is not solved for y. If we did solve it for y, we'd have it in $y-$intercept form. To do that, we would just need to distribute the $\frac{3}{5}$ and add 6 to both sides. That means that the equation of this line in slope-intercept form is $y = \frac{3}{5}x - \frac{6}{5} + 6$, or simply $y = \frac{3}{5}x + \frac{24}{5}$.

Write an Equation Given Two Points

Starting with the slope formula, $m = \frac{y_2-y_1}{x_2-x_1}$, we plug in the $x-$and $y-$values of the two points to get $m = \frac{2-3}{5-(-2)} = \frac{-1}{7}$. We can plug that value of $m$into the point-slope formula to get $y - y_0 = -\frac{1}{7}(x - x_0)$.*For example, suppose we are told that the line passes through the points (-2, 3) and (5, 2). To find the equation of the line, we can start by finding the slope.*

Now we just need to pick one of the two points to plug into the formula. Let's use (5, 2); that gives us $y - 2 = -\frac{1}{7}(x-5)$.

What if we'd picked the other point instead? Then we'd have ended up with the equation $y - 3 = -\frac{1}{7}(x+2)$, which doesn't look the same. That's because there's more than one way to write an equation for a given line in point-slope form. But let's see what happens if we solve each of those equations for y.

Starting with $y - 2 = -\frac{1}{7}(x-5)$, we distribute the $-\frac{1}{7}$ and add 2 to both sides. That gives us $y = -\frac{1}{7}x + \frac{5}{7} + 2$, or $y = -\frac{1}{7}x + \frac{19}{7}$.

On the other hand, if we start with $y - 3 = -\frac{1}{7}(x+2)$, we need to distribute the $-\frac{1}{7}$ and add 3 to both sides. That gives us $y = -\frac{1}{7}x - \frac{2}{7} + 3$, which also simplifies to $y = -\frac{1}{7}x + \frac{19}{7}$.

So whichever point we choose to get an equation in point-slope form, the equation is still mathematically the same, and we can see this when we convert it to $y-$intercept form.

Example 2

A line contains the points (3, 2) and (-2, 4). Write an equation for the line in point-slope form; then write an equation in y−intercept form.

Solution

Find the slope of the line: $m = \frac{y_2-y_1}{x_2-x_1} = \frac{4-2}{-2-3} = -\frac{2}{5}$

Plug in the value of the slope: $y - y_0 = -\frac{2}{5}(x - x_0)$.

Plug point (3, 2) into the equation: $y - 2 = -\frac{2}{5}(x-3)$.

The equation in point-slope form is $y - 2 = -\frac{2}{5}(x-3)$.

To convert to $y-$intercept form, simply solve for y:

$$y-2 = -\frac{2}{5}(x-3) \rightarrow y-2 = -\frac{2}{5}x - \frac{6}{5} \rightarrow y = -\frac{2}{5}x - \frac{6}{5} + 2 = -\frac{2}{5}x + \frac{4}{5}.$$

The equation in $y-$intercept form is $y = -\frac{2}{5}x + \frac{4}{5}$.

Graph an Equation in Point-Slope Form

Another useful thing about point-slope form is that you can use it to graph an equation without having to convert it to slope-intercept form. From the equation $y - y_0 = m(x - x_0)$, you can just read off the slope m and the point (x_0, y_0). To draw the graph, all you have to do is plot the point, and then use the slope to figure out how many units up and over you should move to find another point on the line.

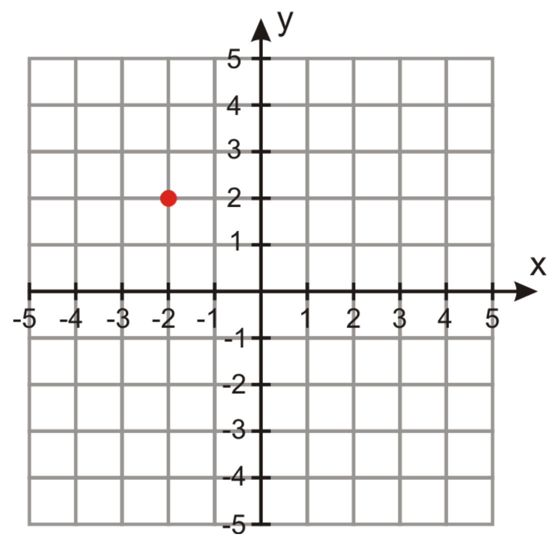
Example 5

Make a graph of the line given by the equation $y + 2 = \frac{2}{3}(x-2)$.

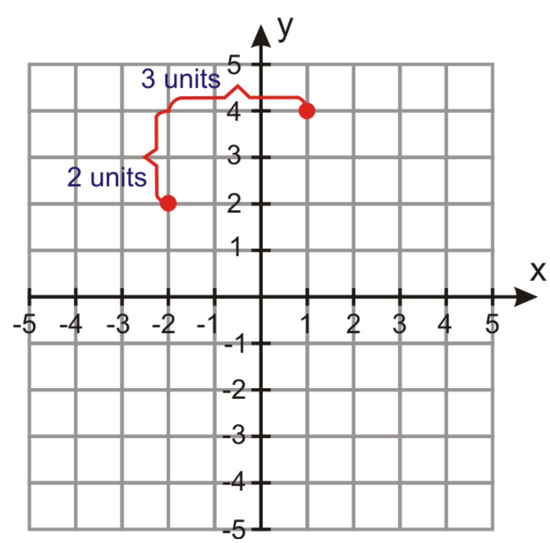
Solution

To read off the right values, we need to rewrite the equation slightly: $y-(-2)=\frac{2}{3}(x-2)$. Now we see that point (2, -2) is on the line and that the slope is $\frac{2}{3}$.

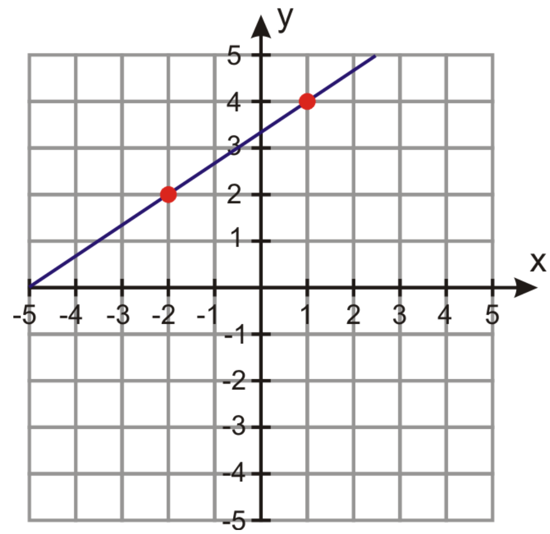
First plot point (2, -2) on the graph:



A slope of $\frac{2}{3}$ tells you that from that point you should move 2 units up and 3 units to the right and draw another point:



Now draw a line through the two points and extend it in both directions:



One useful thing about standard form is that it allows us to write equations for vertical lines, which we can't do in slope-intercept form.You've already encountered another useful form for writing linear equations: **standard form.** An equation in standard form is written $ax+by=c$, where a, b, and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

For example, let's look at the line that passes through points (2, 6) and (2, 9). How would we find an equation for that line in slope-intercept form?

First we'd need to find the slope: $m = \frac{9-6}{0-0} = \frac{3}{0}$. But that slope is undefined because we can't divide by zero. And if we can't find the slope, we can't use point-slope form either.

If we just graph the line, we can see that x equals 2 no matter what y is. There's no way to express that in slope-intercept or point-slope form, but in standard form we can just say that $x+0y=2$, or simply $x=2$.

Converting to Standard Form

To convert an equation from another form to standard form, all you need to do is rewrite the equation so that all the variables are on one side of the equation and the coefficient of x is not negative.

Example 1

Rewrite the following equations in standard form:

a) $y = 5x-7$

b) $y-2 = -3(x+3)$

c) $y = \frac{2}{3}x+\frac{1}{2}$

Solution

We need to rewrite each equation so that all the variables are on one side and the coefficient of x is not negative.

a) $y = 5x-7$

Subtract y from both sides to get $0 = 5x-y-7$.

Add 7 to both sides to get $7 = 5x-y$.

Flip the equation around to put it in standard form: $5x-y = 7$.

b) $y-2 = -3(x+3)$

Distribute the –3 on the right-hand-side to get $y-2 = -3x-9$.

Add $3x$ to both sides to get $y+3x-2=-9$.

Add 2 to both sides to get $y+3x=-7$. Flip that around to get $3x+y=-7$.

c) $y=\frac{2}{3}x+\frac{1}{2}$

Find the common denominator for all terms in the equation –in this case that would be 6.

Multiply all terms in the equation by 6: $6\left(y=\frac{2}{3}x+\frac{1}{2}\right) \Rightarrow 6y=4x+3$

Subtract $6y$ from both sides: $0=4x-6y+3$

Subtract 3 from both sides: $-3=4x-6y$

The equation in standard form is $4x-6y=-3$.

Graphing Equations in Standard Form

When an equation is in slope-intercept form or point-slope form, you can tell right away what the slope is. How do you find the slope when an equation is in standard form?

Well, you could rewrite the equation in slope-intercept form and read off the slope. But there's an even easier way. Let's look at what happens when we rewrite an equation in standard form.

Starting with the equation $ax+by=c$, we would subtract ax from both sides to get $by=-ax+c$. Then we would divide all terms by b and end up with $y=-\frac{a}{b}x+\frac{c}{b}$.

That means that the slope is $-\frac{a}{b}$ and the $y-$intercept is $\frac{c}{b}$. So next time we look at an equation in standard form, we don't have to rewrite it to find the slope; we know the slope is just $-\frac{a}{b}$, where a and b are the coefficients of x and y in the equation.

Example 2

Find the slope and the y−intercept of the following equations written in standard form.

a) $3x+5y=6$

b) $2x-3y=-8$

c) $x-5y=10$

Solution

a) $a=3, b=5$, and $c=6$, so the slope is $-\frac{a}{b}=-\frac{3}{5}$, and the $y-$intercept is $\frac{c}{b}=\frac{6}{5}$.

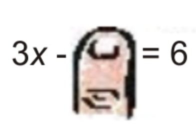
b) $a=2, b=-3$, and $c=-8$, so the slope is $-\frac{a}{b}=\frac{2}{3}$, and the $y-$intercept is $\frac{c}{b}=\frac{8}{3}$.

c) $a=1, b=-5$, and $c=10$, so the slope is $-\frac{a}{b}=\frac{1}{5}$, and the $y-$intercept is $\frac{c}{b}=\frac{10}{-5}=-2$.

Once we've found the slope and $y-$intercept of an equation in standard form, we can graph it easily. But if we start with a graph, how do we find an equation of that line in standard form?

First, remember that we can also use the cover-up method to graph an equation in standard form, by finding the intercepts of the line. For example, let's graph the line given by the equation $3x-2y=6$.

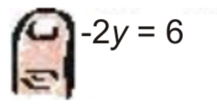
To find the $x-$intercept, cover up the y term (remember, the $x-$intercept is where $y=0$):



$3x=6 \Rightarrow x=2$

The $x-$ intercept is (2, 0).

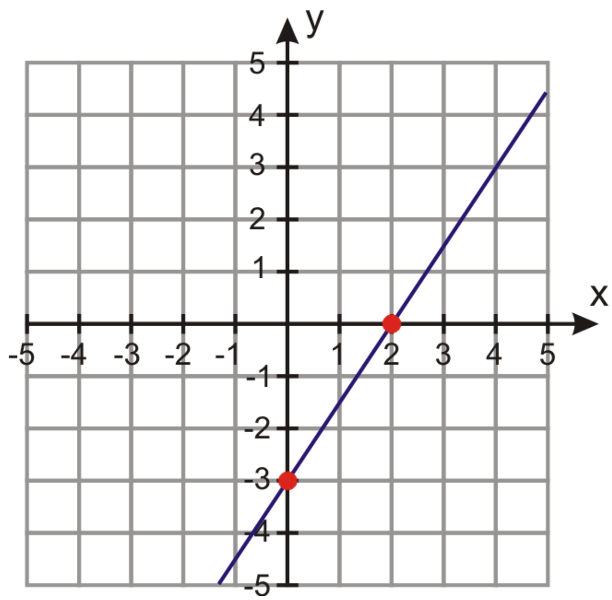
To find the $y-$intercept, cover up the x term (remember, the $y-$intercept is where $x = 0$):



$-2y = 6 \Rightarrow y = -3$

The $y-$intercept is (0, -3).

We plot the intercepts and draw a line through them that extends in both directions:

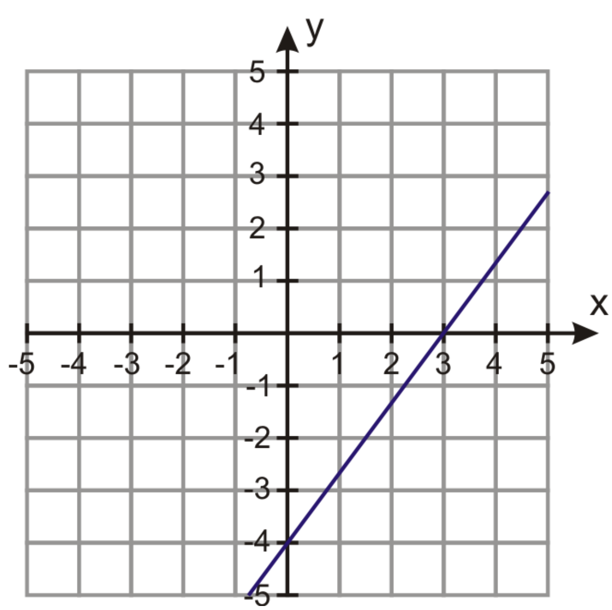


Now we want to apply this process in reverse—to start with the graph of the line and write the equation of the line in standard form.

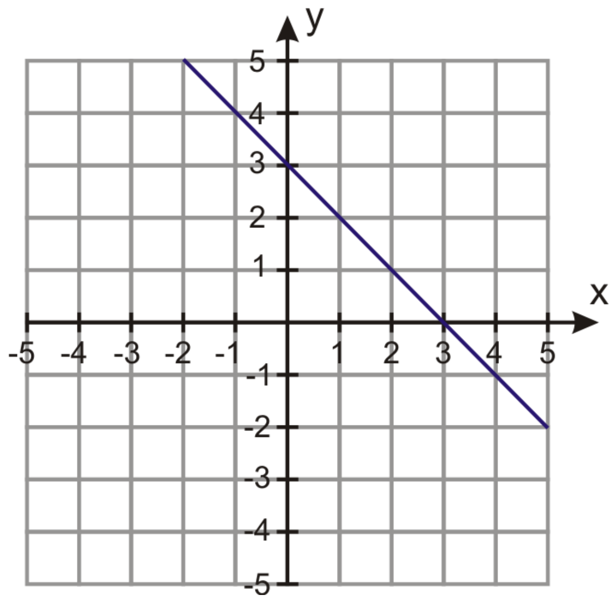
Example 3

Find the equation of each line and write it in standard form.

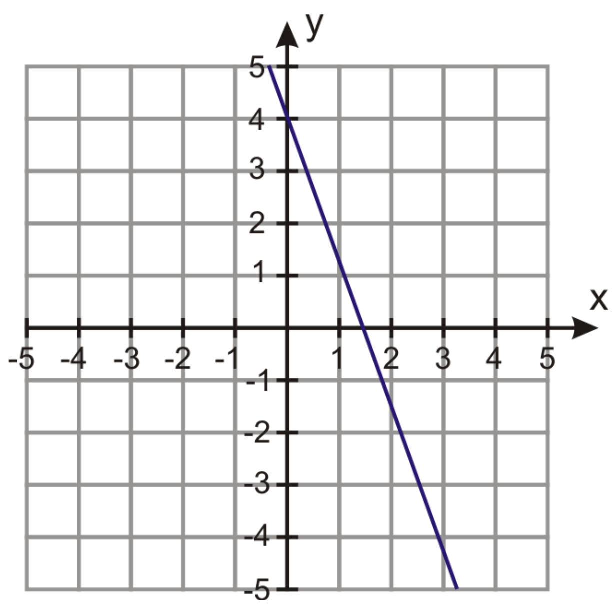
a)



b)



c)



Solution

a) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,-4) \Rightarrow y = -4$

We saw that in standard form $ax+by = c$: if we "cover up" the y term, we get $ax = c$, and if we "cover up" the x term, we get $by = c$.

So we need to find values for a and b so that we can plug in 3 for x and -4 for y and get the same value for c in both cases. This is like finding the least common multiple of the $x-$ and $y-$intercepts.

In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by –3 gives the same result:

$$(x = 3) \times 4 \Rightarrow 4x = 12 \quad \text{and} \quad (y = -4) \times (-3) \Rightarrow -3y = 12$$

Therefore, $a = 4, b = -3$ and $c = 12$ and **the equation in standard form is** $4x - 3y = 12$.

b) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,3) \Rightarrow y = 3$

The values of the intercept equations are already the same, so $a = 1, b = 1$ and $c = 3$. **The equation in standard form is** $x + y = 3$.

c) We see that the $x-$intercept is $\left(\frac{3}{2}, 0\right) \Rightarrow x = \frac{3}{2}$ and the $y-$intercept is $(0,4) \Rightarrow y = 4$

Let's multiply the $x-$intercept equation by $2 \Rightarrow 2x = 3$

Then we see we can multiply the $x-$intercept again by 4 and the $y-$intercept by 3, so we end up with $8x = 12$ and $3y = 12$.

The equation in standard form is $8x + 3y = 12$.

Solving Real-World Problems Using Linear Models in Point-Slope Form

Let's solve some word problems where we need to write the equation of a straight line in point-slope form.

Example 4

Marciel rented a moving truck for the day. Marciel only remembers that the rental truck company charges $40 per day and some number of cents per mile. Marciel drives 46 miles and the final amount of the bill (before tax) is $63. What is the amount per mile the truck rental company charges? Write an equation in point-slope form that describes this situation. How much would it cost to rent this truck if Marciel drove 220 miles?

Solution

Let's define our variables:

$x =$ distance in miles
$y =$ cost of the rental truck

Peter pays a flat fee of $40 for the day; this is the $y-$intercept.

He pays $63 for 46 miles; this is the coordinate point (46,63).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the coordinate point: $63 - y_0 = m(46 - x_0)$

Plug in the point (0, 40): $63 - 40 = m(46 - 0)$

Solve for the slope: $23 = 46m \rightarrow m = \frac{23}{46} = 0.5$

The slope is 0.5 dollars per mile, so the truck company charges 50 cents per mile ($0.5 = 50 cents). Plugging in the slope and the $y-$intercept, the equation of the line is $y = 0.5x + 40$.

To find out the cost of driving the truck 220 miles, we plug in $x = 220$ to get $y - 40 = 0.5(220) \Rightarrow y = \$150$.

Driving 220 miles would cost $150.

Example 5

Anne got a job selling window shades. She receives a monthly base salary and a $6 commission for each window shade she sells. At the end of the month she adds up sales and she figures out that she sold 200 window shades and made $2500. Write an equation in point-slope form that describes this situation. How much is Anne's monthly base salary?

Solution

Let's define our variables:

$x =$ number of window shades sold

$y =$ Annes earnings

We see that we are given the slope and a point on the line:

Nadia gets \$6 for each shade, so the slope is 6.

She made \$2500 when she sold 200 shades, so the point is (200, 2500).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the slope: $y - y_0 = 6(x - x_0)$

Plug in the point (200, 2500): $y - 2500 = 6(x - 200)$

To find Anne's base salary, we plug in $x = 0$ and get $y - 2500 = -1200 \Rightarrow y = \$1300$.

Anne's monthly base salary is \$1300.

Solving Real-World Problems Using Linear Models in Standard Form

Here are two examples of real-world problems where the standard form of an equation is useful.

Example 6

Nadia buys fruit at her local farmer's market. This Saturday, oranges cost \$2 per pound and cherries cost \$3 per pound. She has \$12 to spend on fruit. Write an equation in standard form that describes this situation. If she buys 4 pounds of oranges, how many pounds of cherries can she buy?

Solution

Let's define our variables:

$x =$ pounds of oranges
$y =$ pounds of cherries

The equation that describes this situation is $2x + 3y = 12$.

If she buys 4 pounds of oranges, we can plug $x = 4$ into the equation and solve for y:

$$2(4) + 3y = 12 \Rightarrow 3y = 12 - 8 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3}$$

Nadia can buy $1\frac{1}{3}$ pounds of cherries.

Example 7

Peter skateboards part of the way to school and walks the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If he skateboards for $\frac{1}{2}$ an hour, how long does he need to walk to get to school?

Solution

Let's define our variables:

$x =$ time Peter skateboards
$y =$ time Peter walks

The equation that describes this situation is: $7x + 3y = 6$

If Peter skateboards $\frac{1}{2}$ an hour, we can plug $x = 0.5$ into the equation and solve for y:

$$7(0.5) + 3y = 6 \Rightarrow 3y = 6 - 3.5 \Rightarrow 3y = 2.5 \Rightarrow y = \frac{5}{6}$$

Peter must walk $\frac{5}{6}$ of an hour.

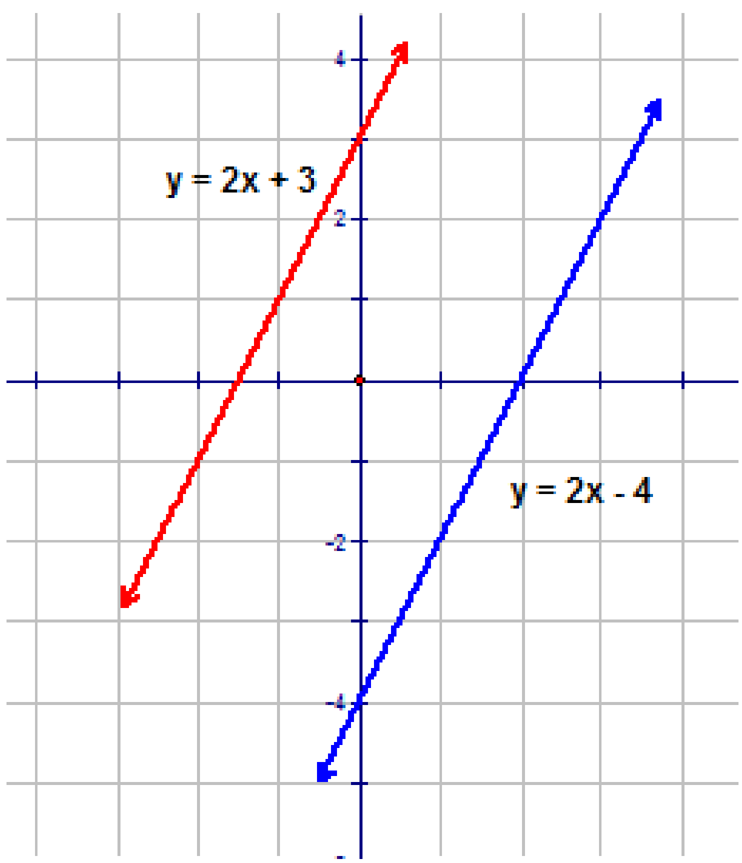
3.7 Equations of Parallel and Perpendicular Lines

Here you will learn about parallel and perpendicular lines and how to determine whether or not two lines are parallel or perpendicular using slope.

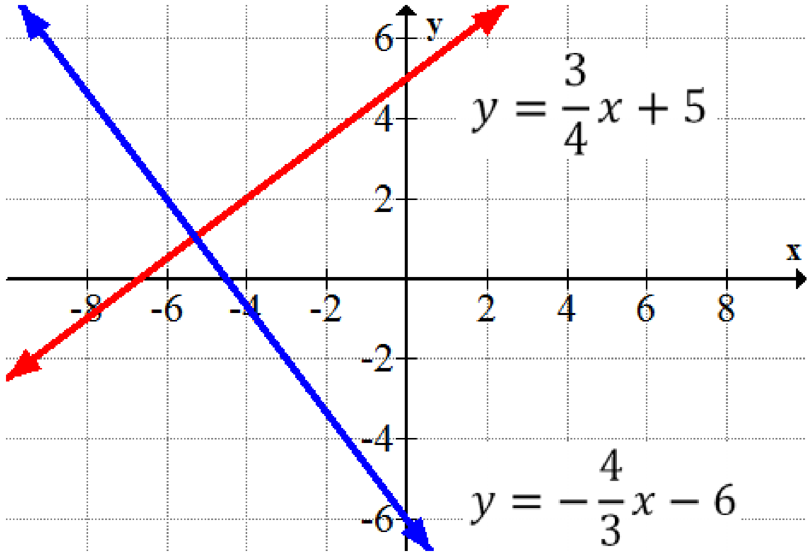
Can you write the equation for the line that passes through the point (–2, –3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

Guidance

Parallel lines are lines in the same plane that never intersect. Parallel lines maintain the **same slope**, or no slope (vertical lines) and the same distance from each other. The following graph shows two lines with the same slope. The slope of each line is 2. Notice that the lines are the same distance apart for the entire length of the lines. The lines will never intersect. The following lines are parallel.



Two lines in the same plane that intersect or cross each other at right angles are **perpendicular lines**. Perpendicular lines have **slopes that are opposite reciprocals**. The following graph shows two lines with slopes that are opposite reciprocals. The slope of one line is $\frac{3}{4}$ and the slope of the other line is $-\frac{4}{3}$. The product of the slopes is negative one. $\left(\frac{3}{4}\right)\left(\frac{-4}{3}\right)=\frac{-12}{12}=-1$. Notice that the lines intersect at a right angle. The lines are perpendicular lines.



You can use the relationship between the slopes of parallel lines and the slopes of perpendicular lines to write the equations of other lines.

Example A

Given the slopes of two lines, tell whether the lines are parallel, perpendicular or neither.

i) $m_1 = 4, m_2 = \frac{1}{4}$

ii) $m_1 = -3, m_2 = \frac{1}{3}$

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$

iv) $m_1 = -1, m_2 = 1$

v) $m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$

Solutions:

i) $m_1 = 4, m_2 = \frac{1}{4}$ The slopes are reciprocals but **not** opposite reciprocals. The lines are neither parallel nor perpendicular.

ii) $m_1 = -3, m_2 = \frac{1}{3}$ The slopes are reciprocals and are also opposite reciprocals. The lines are perpendicular.

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$ The slopes are the same. The fractions are equivalent. The lines are parallel.

iv) $m_1 = -1, m_2 = 1$ The slopes are reciprocals and are also opposite reciprocals. The lines are perpendicular.

v)$m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$ The slopes are not the same. The lines are neither parallel nor perpendicular.

Example B

Determine the equation of the line passing through the point (–4, 6) and parallel to the graph of $3x+2y-7=0$. Write the equation in standard form.

Solution:

If the equation of the line you are looking for is parallel to the given line, then the two lines have the same slope. Begin by expressing $3x+2y-7=0$ in slope-intercept form in order to find its slope.

$$
\begin{aligned}
3x+2y-7&=0\\
3x-3x+2y-7&=0-3x\\
2y-7&=-3x\\
2y-7+7&=-3x+7\\
2y&=-3x+7\\
\frac{\cancel{2}y}{\cancel{2}}&=-\frac{3x}{2}+\frac{7}{2}\\
y&=-\frac{3}{2}x+\frac{7}{2}\\
&\updownarrow\\
y&=\ mx+b
\end{aligned}
$$

The slope of the line is $-\frac{3}{2}$. The line passes through the point $(-4,6)$.

Substitute the values into this equation.

$$
\begin{aligned}
y-y_1&=m(x-x_1)\\
y-6&=\frac{-3}{2}(x--4)\\
y-6&=\frac{-3}{2}(x+4)\\
y-6&=\frac{-3x}{2}-\frac{12}{2}\\
2(y)-2(6)&=\cancel{2}\left(\frac{-3x}{\cancel{2}}\right)-\cancel{2}\left(\frac{12}{\cancel{2}}\right)\\
2y-12&=-3x-12\\
2y-12+12&=-3x-12+12\\
2y&=-3x\\
3x+2y&=-3x+3x\\
3x+2y&=0
\end{aligned}
$$

The equation of the line is

$$\boxed{3x+2y=0}$$

Example C

Determine the equation of the line that passes through the point (6, –2) and is perpendicular to the graph of $3x = 2y-4$. Write the equation in standard form.

Solution: Begin by writing the equation $3x = 2y-4$ in slope-intercept form.

$$\begin{aligned}
3x &= 2y-4\\
2y-4 &= 3x\\
2y-4+4 &= 3x+4\\
2y &= 3x+4\\
\frac{\cancel{2}y}{\cancel{2}} &= \frac{3x}{2}+\frac{4}{2}\\
y &= \frac{3}{2}x+2\\
&\updownarrow\\
y &= mx+b
\end{aligned}$$

The slope of the given line is $\frac{3}{2}$. The slope of the perpendicular line is

$$\boxed{-\frac{2}{3}}$$

. The line passes through the point (6, –2).

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y--2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2x}{3}+\frac{12}{3}\\
3(y)+3(2) &= 3\left(-\frac{2x}{3}\right)+3\left(\frac{12}{3}\right)\\
3(y)+3(2) &= \cancel{3}\left(-\frac{2x}{\cancel{3}}\right)+\cancel{3}\left(\frac{12}{\cancel{3}}\right)\\
3y+6 &= -2x+12\\
3y+6-12 &= -2x+12-12\\
3y-6 &= -2x\\
2x+3y-6 &= -2x+2x\\
2x+3y-6 &= 0
\end{aligned}$$

The equation of the line is

$$\boxed{2x+3y-6=0}$$

.

Concept Problem Revisited

Can you write the equation for the line that passes through the point (–2, –3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the parallel line is the same as the slope of the given line.

$$\begin{aligned} y+2x &= 8 \\ y+2x-2x &= -2x+8 \\ y &= -2x+8 \end{aligned}$$

The slope of the given line is –2. The slope of the parallel line is also

$$\boxed{-2}$$

.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y--3 &= -2(x--2) \\ y+3 &= -2(x+2) \\ y+3 &= -2x-4 \\ y+3 &= -2x-4 \\ 2x+y+3 &= -2x+2x-4 \\ 2x+y+3 &= -4 \\ 2x+y+3+4 &= -4+4 \\ 2x+y+7 &= 0 \end{aligned}$$

The equation of the line is

$$\boxed{2x+y+7=0}$$

Vocabulary

Parallel Lines

Parallel lines are lines in the same plane that have the same slope. The lines never intersect and always maintain the same distance apart.

Perpendicular Lines

Perpendicular lines are lines in the same plane that intersect each other at right angles. The slopes of perpendicular lines are opposite reciprocals. The product of the slopes of two perpendicular lines is –1.

Guided Practice

Determine whether the lines that pass through the two pairs of points are parallel, perpendicular or neither parallel nor perpendicular.

1. (–2, 8), (3, 7) and (4, 3), (9, 2)

2. (2, 5), (8, 7) and (–3, 1), (–2, –2)

3. (4, 6), (–3, –1) and (6, –3), (4, 5)

4. Write the equation for the line that passes through the point (–3, 6) and is perpendicular to the graph of $3x = 5y + 6$. Write the equation of the line in slope-intercept form.

Answers:

1. (–2, 8), (3, 7) and (4, 3), (9, 2)

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Use the formula to calculate the slopes}$$

$$m = \frac{7-8}{3--2} \qquad m = \frac{2-3}{9-4} \qquad \text{Calculate the slopes for each pair of points}$$

$$m = \frac{7-8}{3+2} \qquad \boxed{m = \frac{-1}{5}}$$

$$\boxed{m = \frac{-1}{5}}$$

The slopes of the lines are the same. The lines are parallel.

2. (2, 5), (8, 7) and (–3, 1), (–2, –2)

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Use the formula to calculate the slopes}$$

$$m = \frac{7-5}{8-2} \qquad m = \frac{-2-1}{-2--3} \qquad \text{Calculate the slopes for each pair of points}$$

$$m = \frac{2}{6} \qquad m = \frac{-2-1}{-2+3}$$

$$\boxed{m = \frac{1}{3}} \qquad \boxed{m = \frac{-3}{1}}$$

The slopes of the lines are opposite reciprocals. The lines are perpendicular.

3. (4, 6), (–3, –1) and (6, –3), (4, 5)

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Use the formula to calculate the slopes}$$

$$m = \frac{-1-6}{-3-4} \qquad m = \frac{5--3}{4-6} \qquad \text{Calculate the slopes for each pair of points}$$

$$m = \frac{-7}{-7} \qquad m = \frac{5+3}{4-6}$$

$$\boxed{m = 1} \qquad m = \frac{8}{-2}$$

$$\boxed{m = -4}$$

The lines are neither parallel nor perpendicular.

4. Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the perpendicular line is the opposite reciprocal.

$$\begin{aligned}
3x &= 5y+6 \\
5y+6 &= 3x \\
5y+6-6 &= 3x-6 \\
5y &= 3x-6 \\
\frac{5y}{5} &= \frac{3x}{5}-\frac{6}{5} \\
\frac{\cancel{5}y}{\cancel{5}} &= \frac{3x}{5}-\frac{6}{5} \\
y &= \frac{3}{5}x-\frac{6}{5}
\end{aligned}$$

The slope of the given line is $\frac{3}{5}$. The slope of the perpendicular line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the perpendicular line that passes through the point (–3, 6) is:

$$\begin{aligned}
y &= mx+b \\
6 &= -\frac{5}{3}(-3)+b \\
6 &= -\frac{5}{\cancel{3}}\left(\cancel{-}\overset{-1}{\cancel{3}}\right)+b \\
6 &= 5+b \\
6-5 &= 5-5+b \\
1 &= b
\end{aligned}$$

The y-intercept is (0, 1) and the slope of the line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the line is

$$\boxed{y=-\frac{5}{3}x+1}$$

.

Practice

For each pair of given equations, determine if the lines are parallel, perpendicular or neither.

1. $y=2x-5$ and $y=2x+3$
2. $y=\frac{1}{3}x+5$ and $y=-3x-5$
3. $x=8$ and $x=-2$
4. $y=4x+7$ and $y=-4x-7$

5. $y = -x - 3$ and $y = x + 6$
6. $3y = 9x + 8$ and $y = 3x - 4$

Determine the equation of the line satisfying the following conditions:

7. through the point (5, –6) and parallel to the line $y = 5x + 4$
8. through the point (–1, 7) and perpendicular to the line $y = -4x + 5$
9. containing the point (–1, –5) and parallel to $3x + 2y = 9$
10. containing the point (0, –6) and perpendicular to $6x - 3y + 8 = 0$
11. through the point (2, 4) and perpendicular to the line $y = -\frac{1}{2}x + 3$
12. containing the point (–1, 5) and parallel to $x + 5y = 3$
13. through the point (0, 4) and perpendicular to the line $2x - 5y + 1 = 0$

If $D(4, -1), E(-4, 5)$ and $F(3, 6)$ are the vertices of ΔDEF determine:

14. the equation of the line through D and parallel to EF.
15. the equation of the line containing the altitude from D to EF (the line perpendicular to EF that contains D).

CHAPTER 4

Chapter 4: Functions

Chapter Outline

4.1 Function Notation

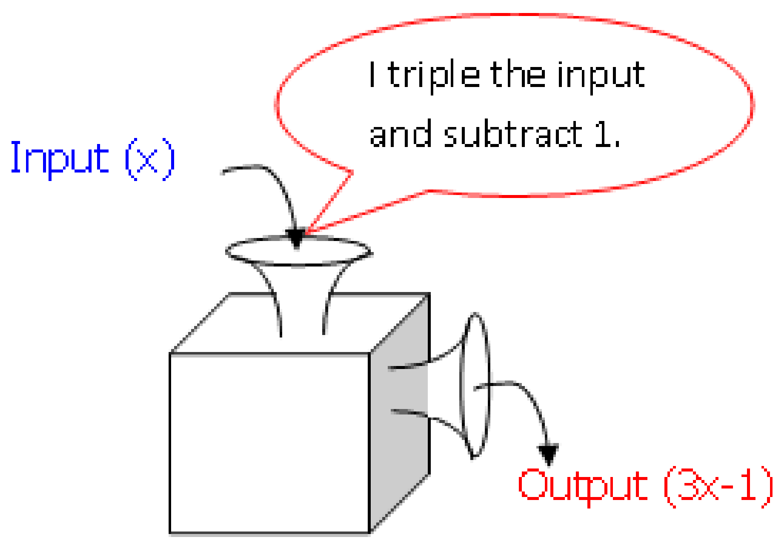
Here you'll learn how to use function notation when working with functions.

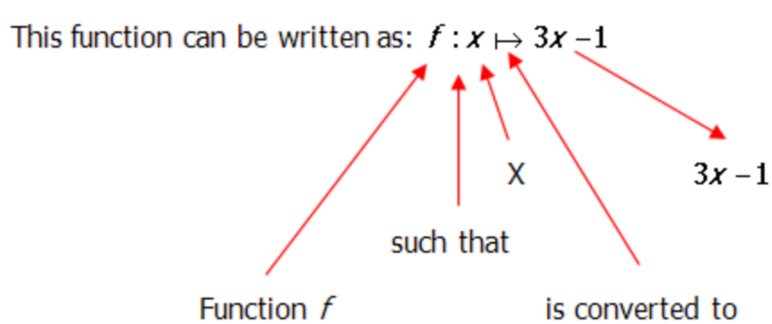
Suppose the value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$.

- Can you determine the value of $V(4)$ and explain what the solution means in the context of this problem?
- Can you determine the value of t when $V(t) = 525$ and explain what this situation represents?
- What was the original cost of the digital camera?

Guidance

A function machine shows how a function responds to an input. If I triple the input and subtract one, the machine will convert x into $3x - 1$. So, for example, if the function is named f, and 3 is fed into the machine, $3(3) - 1 = 8$ comes out.





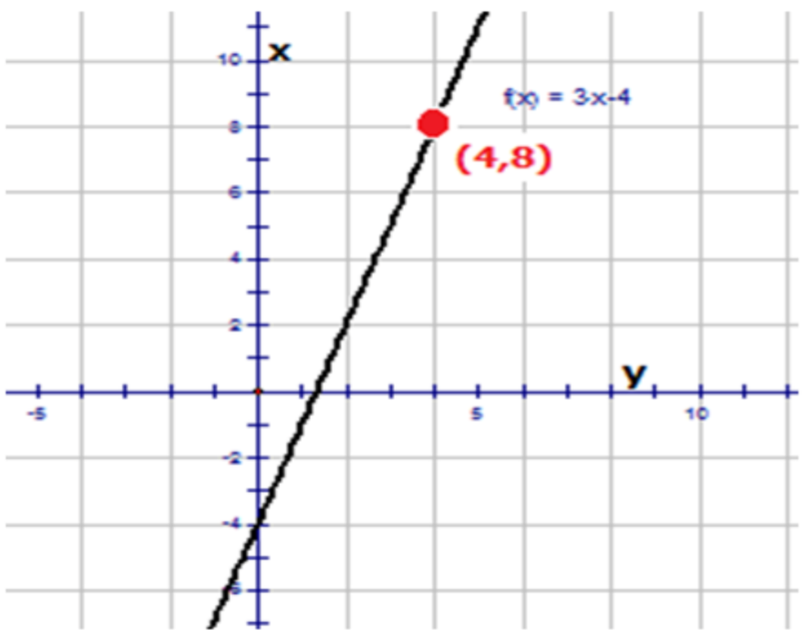
When naming a function the symbol $f(x)$ is often used. The symbol $f(x)$ is pronounced as "f of x." This means that the equation is a function that is written in terms of the variable x. An example of such a function is $f(x) = 3x+4$. Functions can also be written using a letter other than f and a variable other than x. For example, $v(t) = 2t^2 - 5$ and $d(h) = 4h-3$. In addition to representing a function as an equation, you can also represent a function:

- As a graph
- As ordered pairs
- As a table of values
- As an arrow or mapping diagram

When a function is represented as an equation, an ordered pair can be determined by evaluating various values of the assigned variable. Suppose $f(x) = 3x-4$. To calculate $f(4)$, substitute:

$$f(4) = 3(4) - 4$$
$$f(4) = 12 - 4$$
$$f(4) = 8$$

Graphically, if $f(4) = 8$, this means that the point (4, 8) is a point on the graph of the line.



Example A

If $f(x) = x^2 + 2x + 5$ find.

a) $f(2)$

b) $f(-7)$

c) $f(1.4)$

Solution:

To determine the value of the function for the assigned values of the variable, substitute the values into the function.

$$\begin{array}{l} f(x) = x^2 + 2x + 5 \\ \downarrow \quad\quad \downarrow \quad\quad \searrow \\ f(2) = (2)^2 + 2(2) + 5 \\ f(2) = 4 + 4 + 5 \\ \boxed{f(2) = 13} \end{array} \qquad \begin{array}{l} f(x) = x^2 + 2x + 5 \\ \downarrow \quad\quad \downarrow \quad\quad \searrow \\ f(-7) = (-7)^2 + 2(-7) + 5 \\ f(-7) = 49 - 14 + 5 \\ \boxed{f(-7) = 40} \end{array} \qquad \begin{array}{l} f(x) = x^2 + 2x + 5 \\ \downarrow \quad\quad \downarrow \quad\quad \searrow \\ f(1.4) = (1.4)^2 + 2(1.4) + 5 \\ f(1.4) = 1.96 + 2.8 + 5 \\ \boxed{f(1.4) = 9.76} \end{array}$$

Example B

Functions can also be represented as mapping rules. If $g : x \to 5 - 2x$ find the following in simplest form:

a) $g(y)$

b) $g(y-3)$

c) $g(2y)$

Solution:

a) $g(y) = 5 - 2y$

b) $g(y-3) = 5 - 2(y-3) = 5 - 2y + 6 = 11 - 2y$

c) $g(2y) = 5 - 2(2y) = 5 - 4y$

Example C

Let $P(a) = \frac{2a-3}{a+2}$.

a) Evaluate

i) $P(0)$
ii) $P(1)$
iii) $P\left(-\frac{1}{2}\right)$

b) Find a value of 'a' where $P(a)$ does not exist.

c) Find $P(a-2)$ in simplest form

d) Find 'a' if $P(a) = -5$

Solution:

a)

$$\begin{aligned} P(a) &= \frac{2a-3}{a+2} \\ P(0) &= \frac{2(0)-3}{(0)+2} \\ \boxed{P(0) = \frac{-3}{2}} \end{aligned}$$

$$\begin{aligned} P(a) &= \frac{2a-3}{a+2} \\ P(1) &= \frac{2(1)-3}{(1)+2} \\ P(1) &= \frac{2-3}{1+2} \\ \boxed{P(1) = \frac{-1}{3}} \end{aligned}$$

$$\begin{aligned} P(a) &= \frac{2a-3}{a+2} \\ P\left(-\frac{1}{2}\right) &= \frac{2\left(-\frac{1}{2}\right)-3}{\left(-\frac{1}{2}\right)+2} \\ P\left(-\frac{1}{2}\right) &= \frac{{}^{1}\cancel{2}\left(-\frac{1}{\cancel{2}}\right)-3}{-\frac{1}{2}+\frac{4}{2}} \\ P\left(-\frac{1}{2}\right) &= \frac{-1-3}{\frac{3}{2}} \\ P\left(-\frac{1}{2}\right) &= -4 \div \frac{3}{2} \\ P\left(-\frac{1}{2}\right) &= -4\left(\frac{2}{3}\right) \\ \boxed{P\left(-\frac{1}{2}\right)} &= \frac{-8}{3} \end{aligned}$$

b) The function will not exist if the denominator equals zero because division by zero is undefined.

$$\begin{aligned} a+2 &= 0 \\ a+2-2 &= 0-2 \\ &\boxed{a = -2} \end{aligned}$$

Therefore, if $a = -2$, then $P(a) = \frac{2a-3}{a+2}$ does not exist.

c)

$$\begin{aligned} P(a) &= \frac{2a-3}{a+2} \\ P(a-2) &= \frac{2(a-2)-3}{(a-2)+2} && \text{Substitue } a-2 \text{ for } a \\ P(a-2) &= \frac{2a-4-3}{a-2+2} && \text{Remove parentheses} \\ P(a-2) &= \frac{2a-7}{a} && \text{Combine like terms} \\ P(a-2) &= \frac{2\cancel{a}}{\cancel{a}} - \frac{7}{a} && \text{Express the fraction as two separate fractions and reduce.} \\ &\boxed{P(a-2) = 2-\frac{7}{a}} \end{aligned}$$

d)

$$
\begin{aligned}
P(a) &= \frac{2a-3}{a+2} && \\
-5 &= \frac{2a-3}{a+2} && \text{Let } P(a) = -5 \\
-5(a+2) &= \left(\frac{2a-3}{a+2}\right)(a+2) && \text{Multiply both sides by } (a+2) \\
-5a-10 &= \left(\frac{2a-3}{\cancel{a+2}}\right)(\cancel{a+2}) && \text{Simplify} \\
-5a-10 &= 2a-3 && \text{Solve the linear equation} \\
-5a-10-2a &= 2a-2a-3 && \text{Move } 2a \text{ to the left by subtracting} \\
-7a-10 &= -3 && \text{Simplify} \\
-7a-10+10 &= -3+10 && \text{Move 10 to the right side by addition} \\
-7a &= 7 && \text{Simplify} \\
\frac{-7a}{-7} &= \frac{7}{-7} && \text{Divide both sides by -7 to solve for } a. \\
&\boxed{a=-1} &&
\end{aligned}
$$

Concept Problem Revisited

The value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$

- Determine the value of $V(4)$ and explain what the solution means to this problem.
- Determine the value of t when $V(t) = 525$ and explain what this situation represents.
- What was the original cost of the digital camera?

Solution:

- The camera is valued at \$675, 4 years after it was purchased.

$$
\begin{aligned}
V(t) &= 875 - 50t \\
V(4) &= 875 - 50(4) \\
V(4) &= 875 - 200 \\
&\boxed{V(4) = \$675}
\end{aligned}
$$

- The digital camera has a value of \$525, 7 years after it was purchased.

$$
\begin{aligned}
V(t) &= 875 - 50t && \text{Let } V(t) = 525 \\
525 &= 875 - 50t && \text{Solve the equation} \\
525 - 875 &= 875 - 875 - 50t && \\
-350 &= -50t && \\
\frac{-350}{-50} &= \frac{-50t}{-50} && \\
&\boxed{7 = t} &&
\end{aligned}
$$

- The original cost of the camera was \$875.

$$\begin{aligned} V(t) &= 875 - 50t && \text{Let } t = 0. \\ V(0) &= 875 - 50(0) \\ V(0) &= 875 - 0 \\ &\boxed{V(0) = \$875} \end{aligned}$$

Vocabulary

Function
: A ***function*** is a set of ordered pairs (x, y) that shows a relationship where there is only one output for every input. In other words, for every value of x, there is only one value for y.

Guided Practice

1. If $f(x) = 3x^2 - 4x + 6$ find:

i) $f(-3)$
ii) $f(a-2)$

2. If $f(m) = \frac{m+3}{2m-5}$ find 'm' if $f(m) = \frac{12}{13}$

3. The emergency brake cable in a truck parked on a steep hill breaks and the truck rolls down the hill. The distance in feet, d, that the truck rolls is represented by the function $d = f(t) = 0.5t^2$.

i) How far will the truck roll after 9 seconds?
ii) How long will it take the truck to hit a tree which is at the bottom of the hill 600 feet away? *Round your answer to the nearest second.*

Answers:

1. $f(x) = 3x^2 - 4x + 6$

i)

$$\begin{aligned} f(x) &= 3x^2 - 4x + 6 && \text{Substitute } (-3) \text{ for } x \text{ in the function.} \\ f(-3) &= 3(-3)^2 - 4(-3) + 6 && \text{Perform the indicated operations.} \\ f(-3) &= 3(9) + 12 + 6 && \text{Simplify} \\ f(-3) &= 27 + 12 + 6 \\ f(-3) &= 45 \\ &\boxed{f(-3) = 45} \end{aligned}$$

ii)

$$f(x) = 3x^2 - 4x + 6$$
$$f(a-2) = 3(a-2)^2 - 4(a-2) + 6 \qquad \text{Write } (a-2)^2 \text{ in expanded form.}$$
$$f(a-2) = 3(a-2)(a-2) - 4(a-2) + 6 \qquad \text{Perform the indicated operations.}$$
$$f(a-2) = (3a-6)(a-2) - 4(a-2) + 6$$
$$f(a-2) = 3a^2 - 6a - 6a + 12 - 4a + 8 + 6 \qquad \text{Simplify}$$
$$f(a-2) = 3a^2 - 16a + 26$$
$$\boxed{f(a-2) = 3a^2 - 16a + 26}$$

2.

$$f(m) = \frac{m+3}{2m-5}$$
$$\frac{12}{13} = \frac{m+3}{2m-5} \qquad \text{Solve the equation for } m.$$
$$(13)(2m-5)\frac{12}{13} = (13)(2m-5)\frac{m+3}{2m-5}$$
$$\cancel{(13)}(2m-5)\frac{12}{\cancel{13}} = (13)\cancel{(2m-5)}\frac{m+3}{\cancel{2m-5}}$$
$$(2m-5)12 = (13)m+3$$
$$24m - 60 = 13m + 39$$
$$24m - 60 + 60 = 13m + 39 + 60$$
$$24m = 13m + 99$$
$$24m - 13m = 13m - 13m + 99$$
$$11m = 99$$
$$\frac{11m}{11} = \frac{99}{11}$$
$$\frac{\cancel{11}m}{\cancel{11}} = \frac{\overset{9}{\cancel{99}}}{\cancel{11}}$$
$$\boxed{m = 9}$$

3. $d = f(t) = 0.5^2$

i)

$$d = f(t) = 0.5^2 \qquad \text{Substitute 9 for } t.$$
$$f(9) = 0.5(9)^2 \qquad \text{Perform the indicated operations.}$$
$$f(9) = 0.5(81)$$
$$\boxed{f(9) = 40.5 \; feet}$$

After 9 seconds, the truck will roll 40.5 feet.

ii)

$$\begin{aligned} d = f(t) &= 0.5t^2 && \text{Substitute 600 for } d. \\ \color{red}{600} &= 0.5t^2 && \text{Solve for } t. \\ \frac{600}{\color{red}{0.5}} &= \frac{0.5t^2}{\color{red}{0.5}} \\ \frac{\overset{\color{red}{1200}}{\cancel{600}}}{\color{red}{\cancel{0.5}}} &= \frac{\cancel{0.5}t^2}{\color{red}{\cancel{0.5}}} \\ 1200 &= t^2 \\ \sqrt{\color{red}{1200}} &= \sqrt{\color{red}{t^2}} \\ \boxed{34.64 \ seconds \approx t} \end{aligned}$$

The truck will hit the tree in approximately 35 seconds.

Practice

If $g(x) = 4x^2 - 3x + 2$, find expressions for the following:

1. $g(a)$
2. $g(a-1)$
3. $g(a+2)$
4. $g(2a)$
5. $g(-a)$

If $f(y) = 5y - 3$, determine the value of 'y' when:

6. $f(y) = 7$
7. $f(y) = -1$
8. $f(y) = -3$
9. $f(y) = 6$
10. $f(y) = -8$

The value of a Bobby Orr rookie card n years after its purchase is $V(n) = 520 + 28n$.

11. Determine the value of $V(6)$ and explain what the solution means.
12. Determine the value of n when $V(n) = 744$ and explain what this situation represents.
13. Determine the original price of the card.

Let $f(x) = \frac{3x}{x+2}$.

14. When is $f(x)$ undefined?
15. For what value of x does $f(x) = 2.4$?

4.2 Domain and Range

Here you'll learn how to find the domain and range of a relation.

Joseph drove from his summer home to his place of work. To avoid the road construction, Joseph decided to travel the gravel road. After driving for 20 minutes he was 62 miles away from work and after driving for 40 minutes he was 52 miles away from work. Represent the problem on a graph and write a suitable domain and range for the situation.

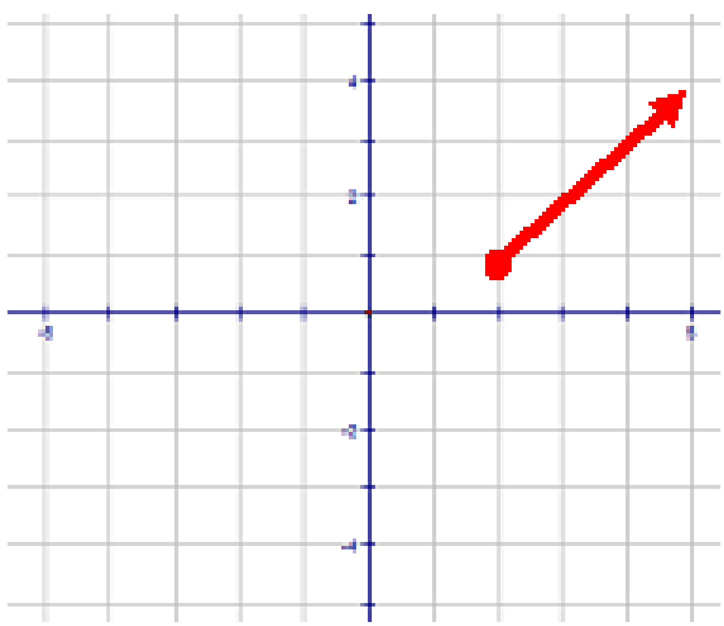
Guidance

The domain of a relation is the set of possible values that 'x' may have. The range of a relation is the set of possible values that 'y' may have. You can write the domain and range of a relation using interval notation and with respect to the number system to which it belongs. Remember:

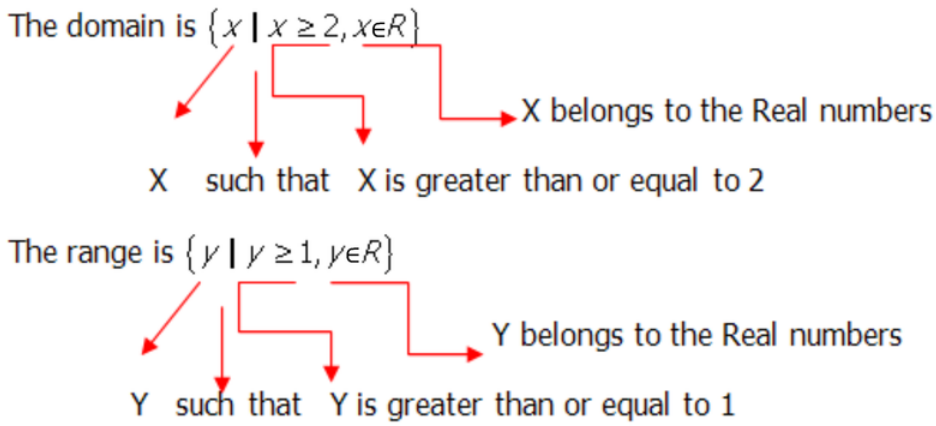
- $Z(\text{integers}) = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $R(\text{real numbers}) = \{\text{all rational and irrational numbers}\}$.

These number systems are very important when the domain and range of a relation are described using interval notation.

A relation is said to be discrete if there are a finite number of data points on its graph. Graphs of discrete relations appear as dots. A relation is said to be continuous if its graph is an unbroken curve with no "holes" or "gaps." The graph of a continuous relation is represented by a line or a curve like the one below. Note that it is possible for a relation to be neither discrete nor continuous.



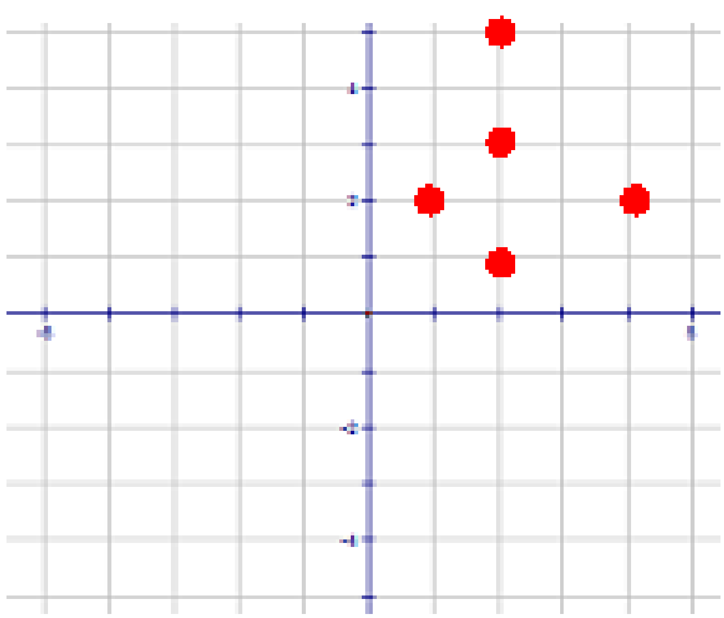
The relation is a straight line that that begins at the point (2, 1). The fact that the points on the line are connected indicates that the relation is continuous. The domain and the range can be written in interval notation, as shown below:



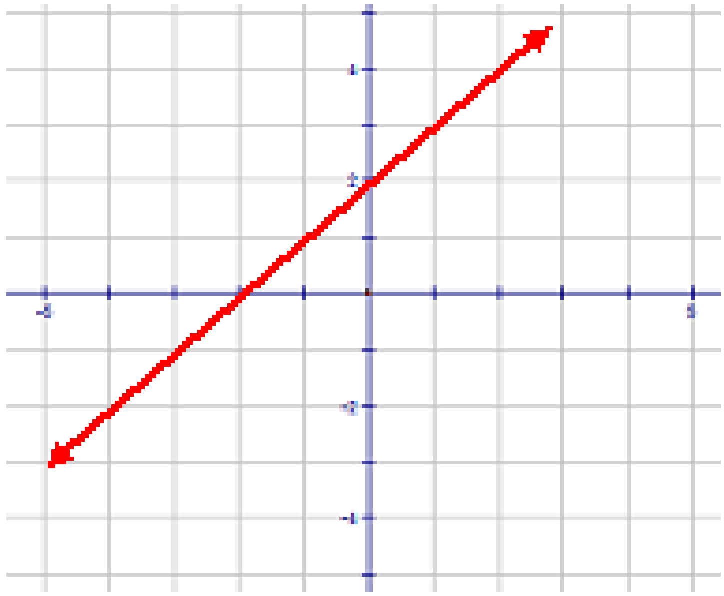
Example A

Which relations are discrete? Which relations are continuous? For each relation, find the domain and range.

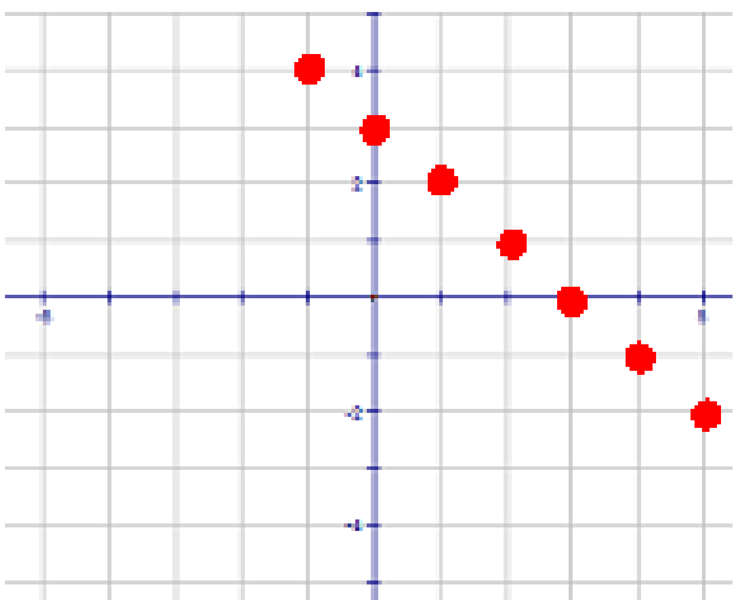
(i)



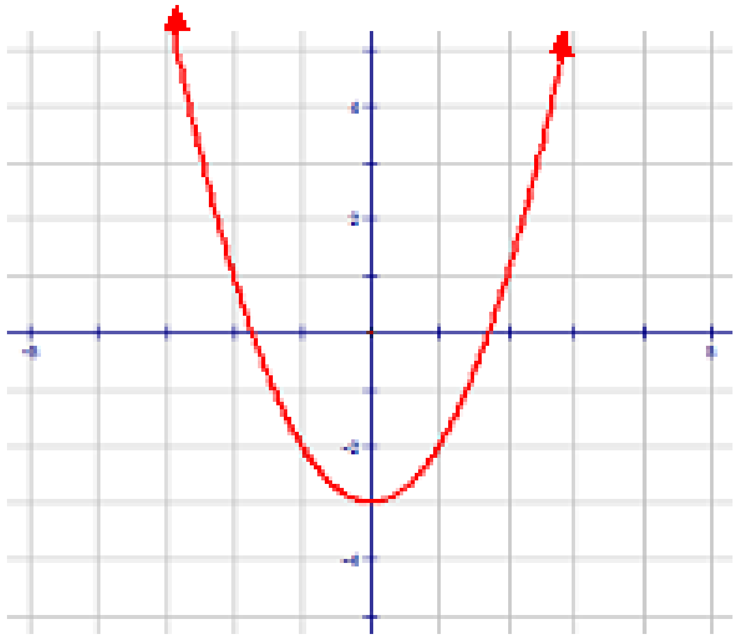
(ii)



(iii)



(iv)



Solution:

(i) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{1,2,4\}$. The range is $\{1,2,3,5\}$

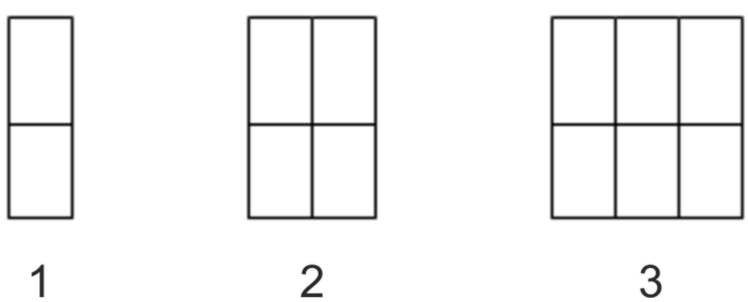
(ii) The graph appears as a straight line. Therefore, the relation is continuous. $D = \{x|x \in R\}$ $R = \{y|y \in R\}$

(iii) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{-1,0,1,2,3,4,5\}$. The range is $\{-2,-1,0,1,2,3,4\}$

(iv) The graph appears as a curve. Therefore, the relation is continuous. $D = \{x|x \in R\}$ $R = \{y|y \geq -3, y \in R\}$

Example B

Whether a relation is discrete, continuous, or neither can often be determined without a graph. The domain and range can be determined without a graph as well. Examine the following toothpick pattern.



Complete the table below to determine the number of toothpicks needed for the pattern.

TABLE 4.1:

Pattern number (n)	1	2	3	4	5	...	n	...	200
Number of toothpicks (t)									

Is the data continuous or discrete? Why?

What is the domain?

What is the range?

Solution:

TABLE 4.2:

Pattern number (n)	1	2	3	4	5	...	n	...	200
Number of toothpicks (t)	7	12	17	22	27		$5n+2$		1002

The number of toothpicks in any pattern number is the result of multiplying the pattern number by 5 and adding 2 to the product.

The number of toothpicks in pattern number 200 is:

$$\begin{aligned}
t &= 5n+2 \\
t &= 5(200)+2 \\
t &= 1000+2 \\
t &= 1002
\end{aligned}$$

The data must be discrete. The graph would be dots representing the pattern number and the number of toothpicks. It is impossible to have a pattern number or a number of toothpicks that are not natural numbers. Therefore, the points would not be joined.

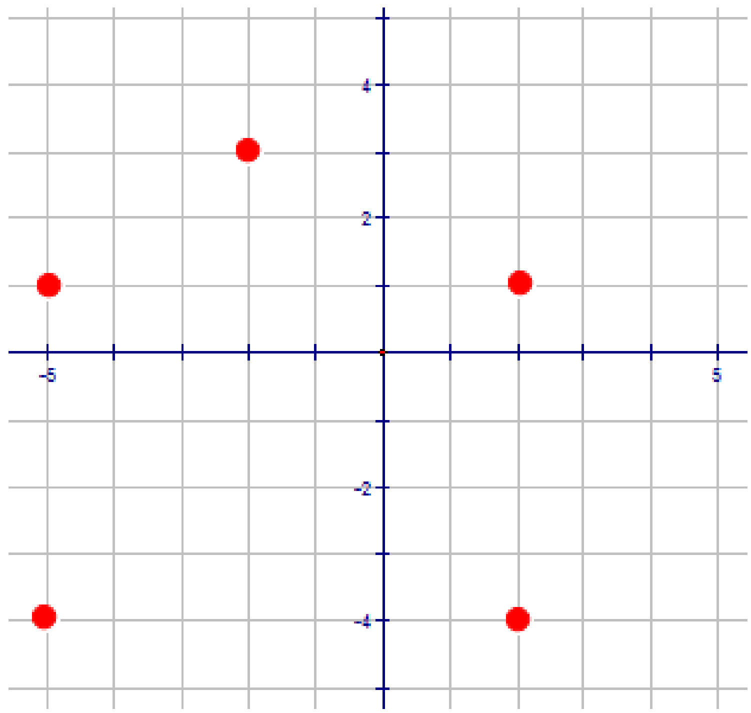
The domain and range are:

$$D = \{x|x \in N\} \quad R = \{y|y = 5x+2, x \in N\}$$

If the range is written in terms of a function, then the number system to which x belongs must be designated in the range.

Example C

Can you state the domain and the range of the following relation?



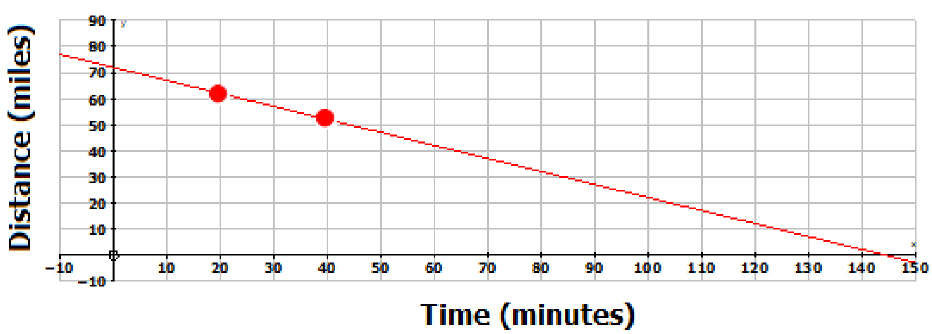
Solution:

The points indicated on the graph are $\{(-5,-4),(-5,1),(-2,3),(2,1),(2,-4)\}$

The domain is $\{-5,-2,2\}$ and the range is $\{-4,1,3\}$.

Concept Problem Revisited

Joseph drove from his summer home to his place of work. To avoid the road construction, Joseph decided to travel the gravel road. After driving for 20 minutes he was 62 miles away from work and after driving for 40 minutes he was 52 miles away from work. Represent the problem on a graph and write a suitable domain and range for the situation.



To represent the problem on a graph, plot the points (20, 62) and (40, 52). The points can be joined with a straight line since the data is continuous. The distance traveled changes continuously as the time driving changes. The y-intercept represents the distance from Joseph's summer home to his place of work. This distance is approximately 72 miles. The x-intercept represents the time it took Joseph to drive from his summer home to work. This time is approximately 145 minutes.

Time cannot be a negative quantity. Therefore, the smallest value for the number of minutes would have to be zero. This represents the time Joseph began his trip. A suitable domain for this problem is $D = \{x|0 \le x \le 145, x \in R\}$

The distance from his summer home to work cannot be a negative quantity. This distance is represented on the y-axis as the y-intercept and is the distance before he begins to drive. A suitable range for the problem is $R = \{y|0 \le y \le 72, y \in R\}$

The domain and range often depend on the quantities presented in the problem. In the above problem, the quantities of time and distance could not be negative. As a result, the values of the domain and the range had to be positive.

Vocabulary

Continuous
A relation is said to be ***continuous*** if it is an unbroken curve with no "holes" or "gaps".

Discrete
A relation is said to be ***discrete*** if there are a finite number of data points on its graph. Graphs of ***discrete*** relations appear as dots.

Domain
The ***domain*** of a relation is the set of possible values that 'x' may have.

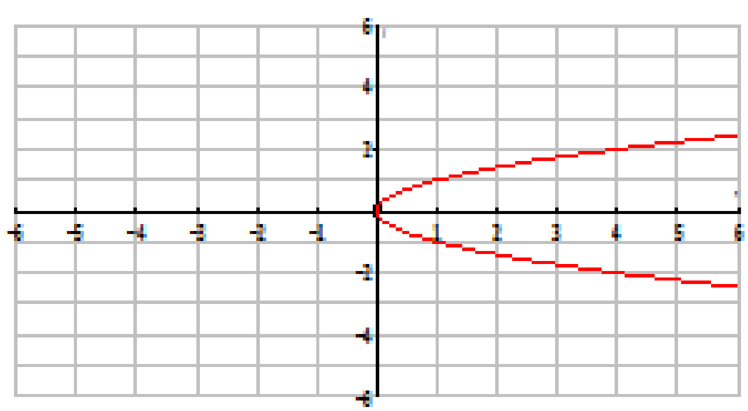
Range
The ***range*** of a relation is the set of possible values that 'y' may have.

Coordinates
The ***coordinates*** are the ordered pair (x, y) that represents a point on the Cartesian plane.

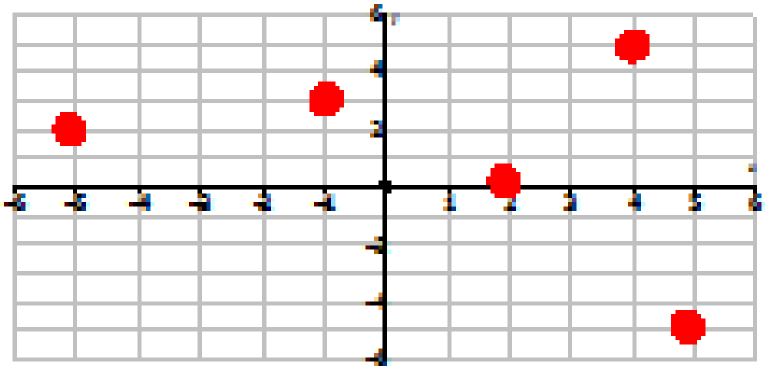
Guided Practice

1. Which relation is discrete? Which relation is continuous?

(i)

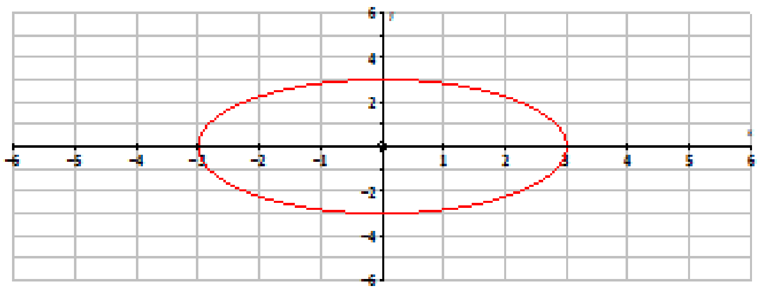


(ii)

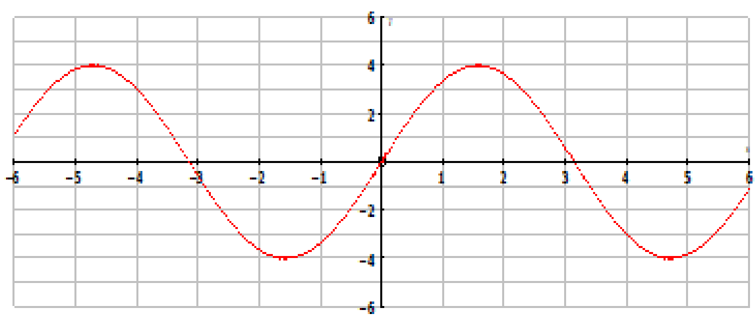


2. State the domain and the range for each of the following relations:

(i)



(ii)



3. A computer salesman's wage consists of a monthly salary of $200 plus a bonus of $100 for each computer sold.

(a) Complete the following table of values:

TABLE 4.3:

Number of computers sold	**0**	**2**	**5**	**10**	**18**
Wages in dollars for the month ($)					

(b) Sketch the graph to represent the monthly salary ($), against the number (N), of computers sold.
(c) Use the graph to write a suitable domain and range for the problem.

Answers:

1. (i) The graph clearly shows that the points are joined. Therefore the data is continuous.

(ii) The graph shows the plotted points as dots that are not joined. Therefore the data is discrete.

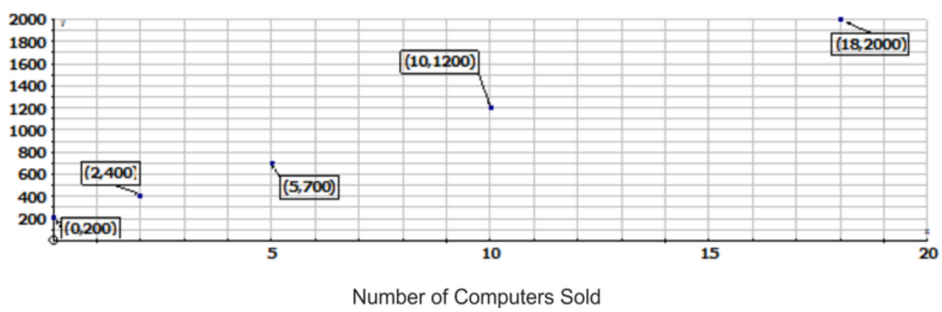
2. (i) The domain represents the values of 'x'. $D = \{x|-3 \leq x \leq 3, x \in R\}$

The range represents the values of 'y'. $R = \{y|-3 \leq y \leq 3, y \in R\}$
(ii) $D = \{x|x \in R\}$
$R = \{y|-4 \leq y \leq 4, y \in R\}$

3.

TABLE 4.4:

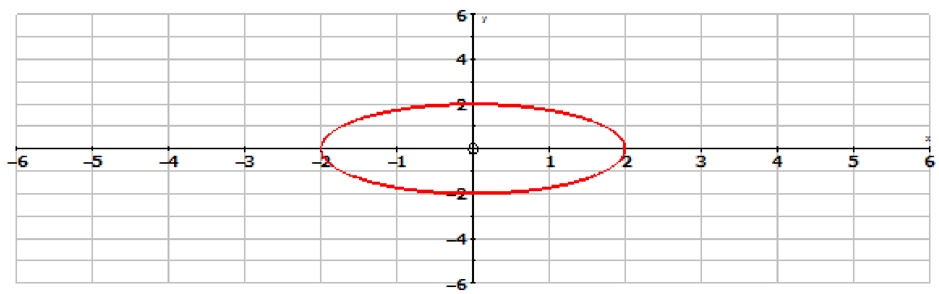
Number of computers sold	0	2	5	10	18
Wages in dollars for the month ($)	$200	$400	$700	$1200	$2000



(c) The graph shows that the data is discrete. (The salesman can't sell a portion of a computer, so the data points can't be connected.) The number of computers sold and must be whole numbers. The wages must be natural numbers.
A suitable domain is $D = \{x|x \geq 0, x \in W\}$
A suitable domain is $R = \{y|y = 200 + 100x, x \in N\}$

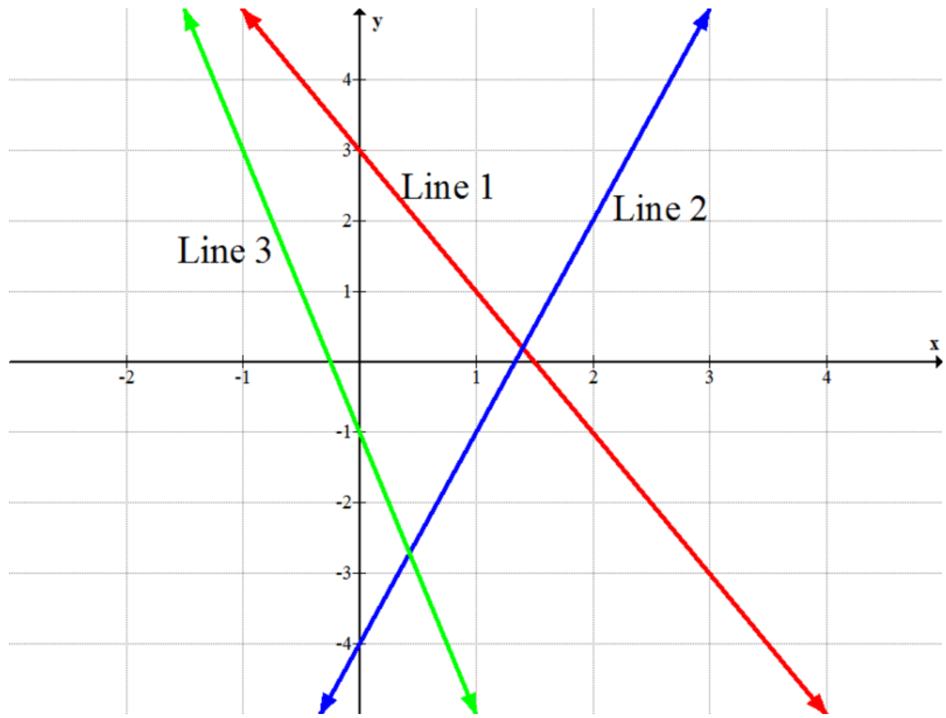
Practice

Use the graph below for #1 and #2.



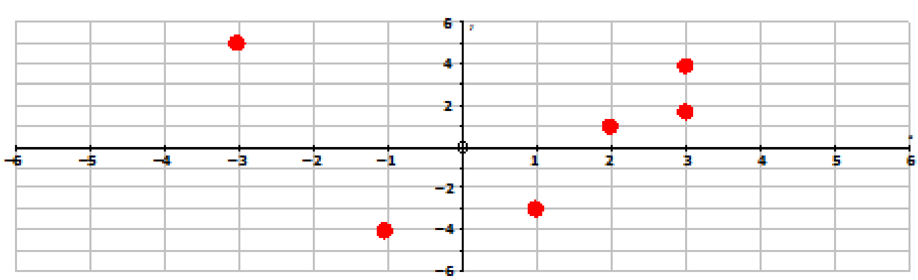
1. Is the relation discrete, continuous, or neither?
2. Find the domain and range for the relation.

Use the graph below for #3 and #4.



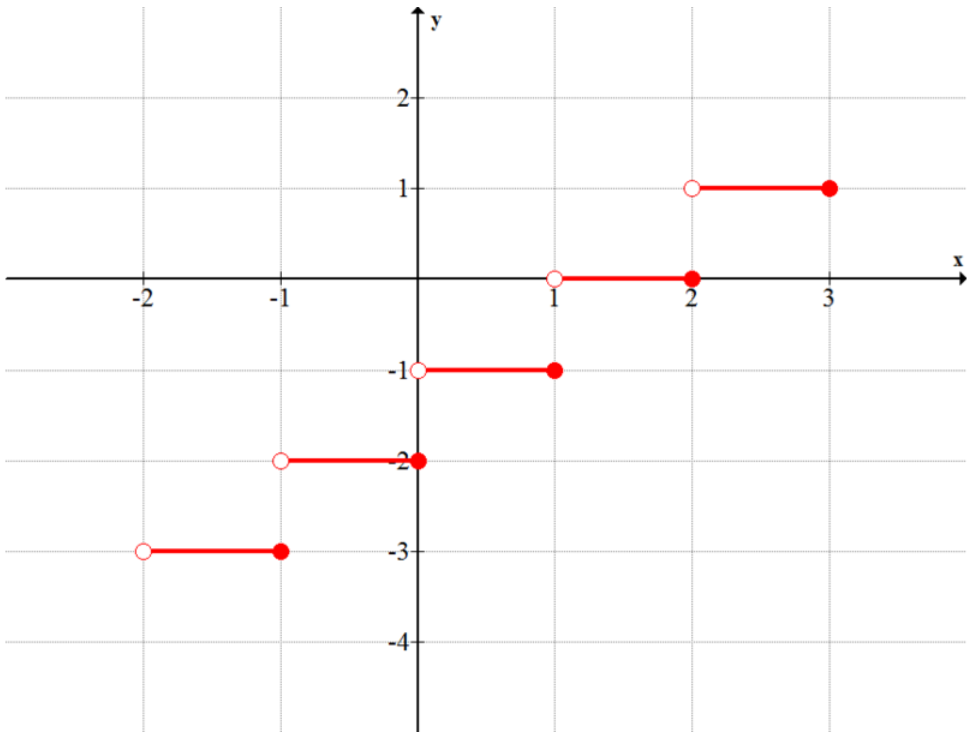
3. Is the relation discrete, continuous, or neither?
4. Find the domain and range for each of the three relations.

Use the graph below for #5 and #6.



5. Is the relation discrete, continuous, or neither?
6. Find the domain and range for the relation.

Use the graph below for #7 and #8.



7. Is the relation discrete, continuous, or neither?
8. Find the domain and range for the relation.

Examine the following pattern.

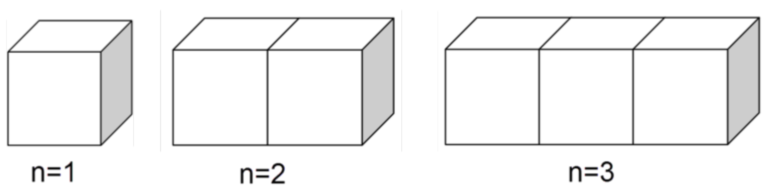


TABLE 4.5:

Number of Cubes (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**200**
Number of visible faces (f)	6	10	14						

9. Complete the table below the pattern.
10. Is the relation discrete, continuous, or neither?
11. Write a suitable domain and range for the pattern.

Examine the following pattern.

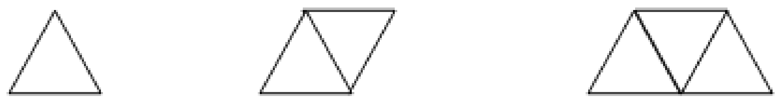


TABLE 4.6:

Number of triangles (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**100**
Number of toothpicks (t)									

12. Complete the table below the pattern.
13. Is the relation discrete, continuous, or neither?
14. Write a suitable domain and range for the pattern.

Examine the following pattern.

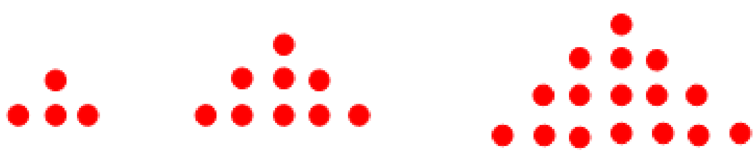


TABLE 4.7:

Pattern Number (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**100**
Number of dots (d)									

15. Complete the table below the pattern.
16. Is the relation discrete, continuous, or neither?
17. Write a suitable domain and range for the pattern.

4.3 Graphs of Functions based on Rules

Here you'll learn how to graph a function from a given rule.

What if you were given a function rule like $f(x) = \sqrt{2x^2 + 1}$. How could you graph that function? After completing this Concept, you'll be able to create a table of values to graph functions like this one in the coordinate plane.

Guidance

We can always make a graph from a function rule, by substituting values in for the variable and getting the corresponding output value.

Example A

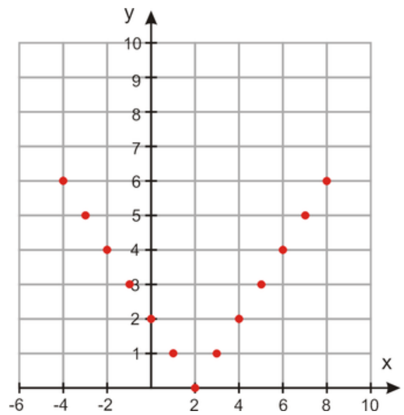
Graph the following function: $f(x) = |x - 2|$

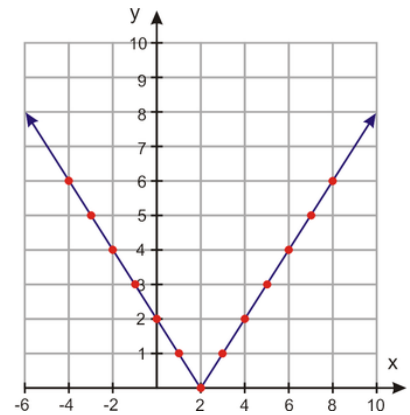
Solution

Make a table of values. Pick a variety of negative and positive values for x. Use the function rule to find the value of y for each value of x. Then, graph each of the coordinate points.

TABLE 4.8:

x	$y = f(x) = \mid x - 2 \mid$
-4	$\mid -4-2 \mid = \mid -6 \mid = 6$
-3	$\mid -3-2 \mid = \mid -5 \mid = 5$
-2	$\mid -2-2 \mid = \mid -4 \mid = 4$
-1	$\mid -1-2 \mid = \mid -3 \mid = 3$
0	$\mid 0-2 \mid = \mid -2 \mid = 2$
1	$\mid 1-2 \mid = \mid -1 \mid = 1$
2	$\mid 2-2 \mid = \mid 0 \mid = 0$
3	$\mid 3-2 \mid = \mid 1 \mid = 1$
4	$\mid 4-2 \mid = \mid 2 \mid = 2$
5	$\mid 5-2 \mid = \mid 3 \mid = 3$
6	$\mid 6-2 \mid = \mid 4 \mid = 4$
7	$\mid 7-2 \mid = \mid 5 \mid = 5$
8	$\mid 8-2 \mid = \mid 6 \mid = 6$





It is wise to work with a lot of values when you begin graphing. As you learn about different types of functions, you will start to only need a few points in the table of values to create an accurate graph.

Example B

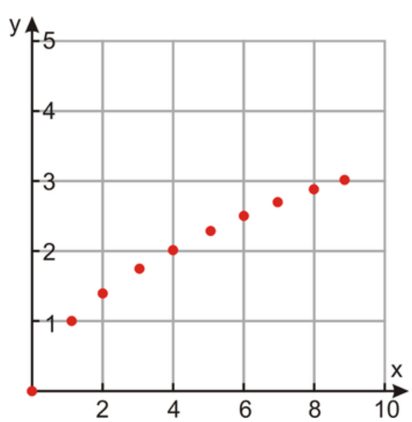
Graph the following function: $f(x) = \sqrt{x}$

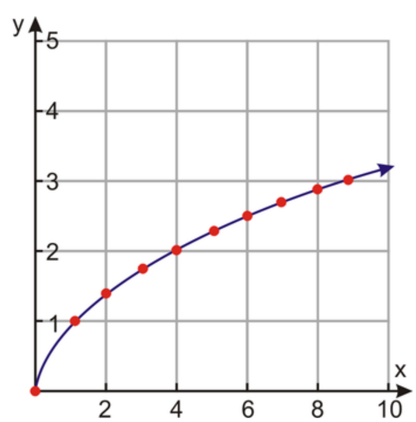
Solution

Make a table of values. We know x can't be negative because we can't take the square root of a negative number. The domain is all positive real numbers, so we pick a variety of positive integer values for x. Use the function rule to find the value of y for each value of x.

TABLE 4.9:

x	$y = f(x) = \sqrt{x}$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	$\sqrt{4} = 2$
5	$\sqrt{5} \approx 2.24$
6	$\sqrt{6} \approx 2.45$
7	$\sqrt{7} \approx 2.65$
8	$\sqrt{8} \approx 2.83$
9	$\sqrt{9} = 3$





Note that the range is all positive real numbers.

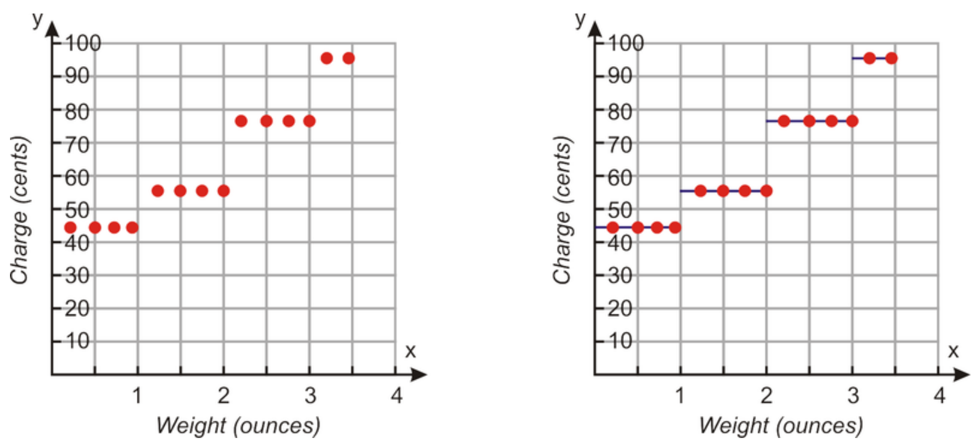
Example C

The post office charges 41 cents to send a letter that is one ounce or less and an extra 17 cents for each additional ounce or fraction of an ounce. This rate applies to letters up to 3.5 ounces.

Solution

Make a table of values. We can't use negative numbers for x because it doesn't make sense to have negative weight. We pick a variety of positive values for x, making sure to include some decimal values because prices can be decimals too. Then we use the function rule to find the value of y for each value of x.

x	0	0.2	0.5	0.8	1	1.2	1.5	1.8	2	2.2	2.5	2.8	3	3.2	3.5
y	0	41	41	41	41	58	58	58	58	75	75	75	75	92	92



Guided Practice

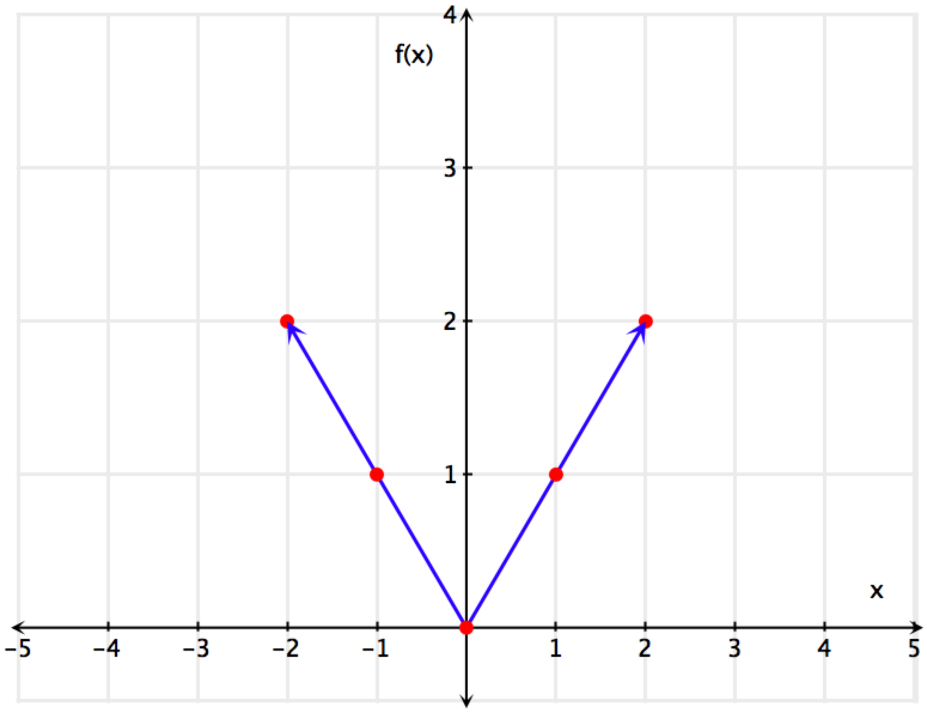
Graph the following function: $f(x) = \sqrt{x^2}$

Solution

Make a table of values. Even though x can't be negative inside the square root, because we are squaring x first, the domain is all real numbers. So we integer values for x which are on either side of zero. Use the function rule to find the value of y for each value of x.

TABLE 4.10:

x	$y = f(x) = \sqrt{x^2}$
-2	$\sqrt{(-2)^2} = 2$
-1	$\sqrt{(-1)^2} = 1$
0	$\sqrt{0^2} = 0$
1	$\sqrt{1^2} = 1$
2	$\sqrt{2^2} = 2$



Note that the range is all positive real numbers, and that this looks like an absolute value function.

Practice

Graph the following functions.

1. Vanson spends $20 a month on his cat.
2. Brandon is a member of a movie club. He pays a $50 annual membership and $8 per movie.
3. $f(x) = (x-2)^2$
4. $f(x) = 3.2^x$
5. $f(t) = 27t - t^2$
6. $f(w) = \frac{w}{4} + 5$
7. $f(x) = t + 2t^2 + 3t^3$
8. $f(x) = (x-1)(x+3)$
9. $f(x) = \frac{x}{3} + \frac{x^2}{5}$
10. $f(x) = \sqrt{2x}$

4.4 Linear Interpolation and Extrapolation

Here you'll learn how to use linear interpolation to fill in gaps in data and linear extrapolation to estimate values outside a data set's range.

What if you were given a table of values that showed the average lifespan for Americans for each decade from 1950 to 2010? How could you use that data to estimate the average lifespan in 2020 or 2030? After completing this Concept, you'll be able to make predictions from linear models like this one.

Linear Interpolation

We use linear interpolation to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.

Example A

The following table shows the median ages of first marriage for men and women, as gathered by the U.S. Census Bureau.

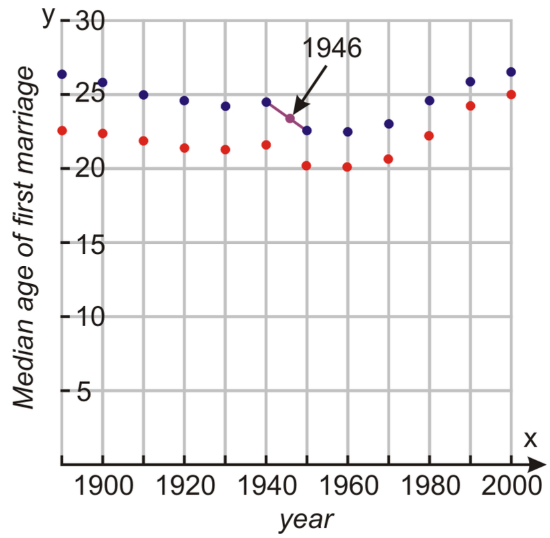
TABLE 4.11:

Year	Median age of males	Median age of females
1890	26.1	22.0
1900	25.9	21.9
1910	25.1	21.6
1920	24.6	21.2
1930	24.3	21.3
1940	24.3	21.5
1950	22.8	20.3
1960	22.8	20.3
1970	23.2	20.8
1980	24.7	22.0
1990	26.1	23.9
2000	26.8	25.1

Estimate the median age for the first marriage of a male in the year 1946.

Solution

We connect the two points on either side of 1946 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$\begin{aligned}
m &= \frac{22.8-24.3}{1950-1940} = \frac{-1.5}{10} = -0.15 \\
y &= -0.15x+b \\
24.3 &= -0.15(1940)+b \\
b &= 315.3
\end{aligned}$$

Our equation is $y = -0.15x + 315.3$.

To estimate the median age of marriage of males in the year 1946, we plug $x = 1946$ into the equation we just found:

$y = -0.15(1946) + 315.3 = 23.4$ **years old**

For non-linear data, linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation,** which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.

Linear Extrapolation

Linear extrapolation can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Example B

The winning times for the women's 100 meter race are given in the following table. Estimate the winning time in the year 2010. Is this a good estimate?

TABLE 4.12:

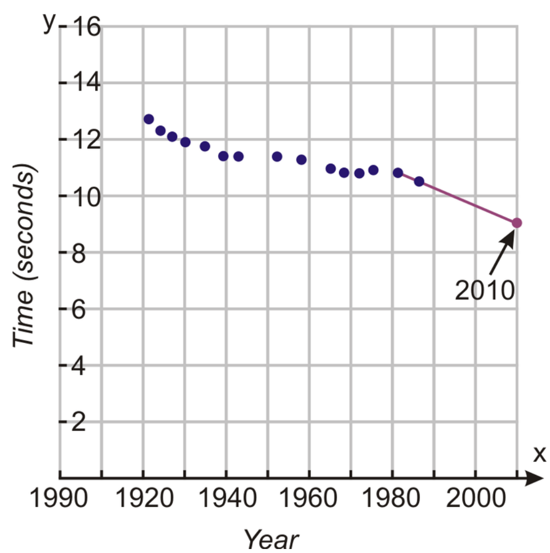
Winner	Country	Year	Time (seconds)
Mary Lines	UK	1922	12.8
Leni Schmidt	Germany	1925	12.4
Gerturd Glasitsch	Germany	1927	12.1
Tollien Schuurman	Netherlands	1930	12.0

TABLE 4.12: (continued)

Winner	Country	Year	Time (seconds)
Helen Stephens	USA	1935	11.8
Lulu Mae Hymes	USA	1939	11.5
Fanny Blankers-Koen	Netherlands	1943	11.5
Marjorie Jackson	Australia	1952	11.4
Vera Krepkina	Soviet Union	1958	11.3
Wyomia Tyus	USA	1964	11.2
Barbara Ferrell	USA	1968	11.1
Ellen Strophal	East Germany	1972	11.0
Inge Helten	West Germany	1976	11.0
Marlies Gohr	East Germany	1982	10.9
Florence Griffith Joyner	USA	1988	10.5

Solution

We start by making a scatter plot of the data; then we connect the last two points on the graph and find the equation of the line.



$$
\begin{aligned}
m &= \frac{10.5-10.9}{1988-1982} = \frac{-0.4}{6} = -0.067 \\
y &= -0.067x + b \\
10.5 &= -0.067(1988) + b \\
b &= 143.7
\end{aligned}
$$

Our equation is $y = -0.067x + 143.7$.

The winning time in year 2010 is estimated to be:

$y = -0.067(2010) + 143.7 = 9.03$ **seconds.**

Unfortunately, this estimate actually isn't very accurate. This example demonstrates the weakness of linear extrapolation; it uses only a couple of points, instead of using *all* the points like the best fit line method, so it doesn't give as accurate results when the data points follow a linear pattern. In this particular example, the last data point clearly doesn't fit in with the general trend of the data, so the slope of the extrapolation line is much steeper than it would be if we'd used a line of best fit. (As a historical note, the last data point corresponds to the winning time for Florence Griffith Joyner in 1988. After her race she was accused of using performance-enhancing drugs, but this fact was never proven. In addition, there was a question about the accuracy of the timing: some officials said that tail-wind was not accounted for in this race, even though all the other races of the day were affected by a strong wind.)

Here's an example of a problem where linear extrapolation does work better than the line of best fit method.

Example C

A cylinder is filled with water to a height of 73 centimeters. The water is drained through a hole in the bottom of the cylinder and measurements are taken at 2 second intervals. The following table shows the height of the water level in the cylinder at different times.

TABLE 4.13:

Time (seconds)	Water level (cm)
0.0	73
2.0	63.9
4.0	55.5
6.0	47.2
8.0	40.0
10.0	33.4
12.0	27.4
14.0	21.9
16.0	17.1
18.0	12.9
20.0	9.4
22.0	6.3
24.0	3.9
26.0	2.0
28.0	0.7
30.0	0.1

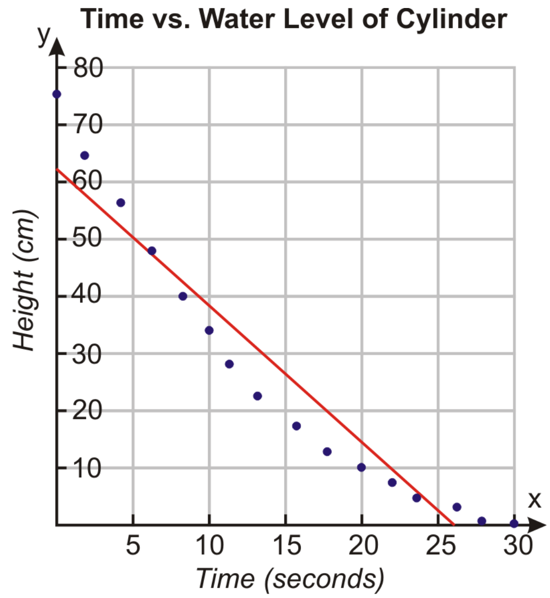
a) *Find the water level at time 15 seconds.*

b) *Find the water level at time 27 seconds*

c) *What would be the original height of the water in the cylinder if the water takes 5 extra seconds to drain? (Find the height at time of –5 seconds.)*

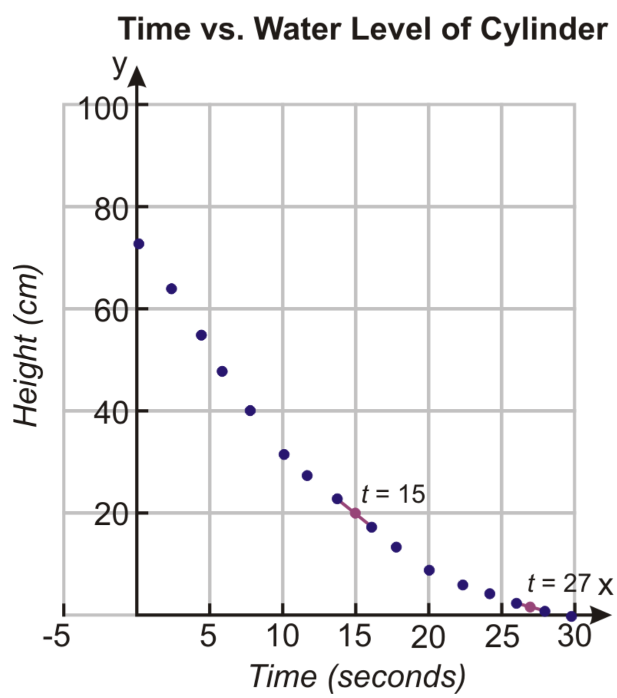
Solution

Here's what the line of best fit would look like for this data set:



Notice that the data points don't really make a line, and so the line of best fit still isn't a terribly good fit. Just a glance tells us that we'd estimate the water level at 15 seconds to be about 27 cm, which is *more* than the water level at 14 seconds. That's clearly not possible! Similarly, at 27 seconds we'd estimate the water to have all drained out, which it clearly hasn't yet.

So let's see what happens if we use linear extrapolation and interpolation instead. First, here are the lines we'd use to interpolate between 14 and 16 seconds, and between 26 and 28 seconds.



a) The slope of the line between points (14, 21.9) and (16, 17.1) is $m = \frac{17.1-21.9}{16-14} = \frac{-4.8}{2} = -2.4$. So $y = -2.4x + b \Rightarrow 21.9 = -2.4(14) + b \Rightarrow b = 55.5$, and the equation is $y = -2.4x + 55.5$.

Plugging in $x = 15$ gives us $y = -2.4(15) + 55.5 = 19.5\ cm$.

b) The slope of the line between points (26, 2) and (28, 0.7) is $m = \frac{0.7-2}{28-26} = \frac{-1.3}{2} = -.65$, so $y = -.65x + b \Rightarrow 2 = -.65(26) + b \Rightarrow b = 18.9$, and the equation is $y = -.65x + 18.9$.

Plugging in $x = 27$, we get $y = -.65(27) + 18.9 = 1.35\ cm$.

c) Finally, we can use extrapolation to estimate the height of the water at -5 seconds. The slope of the line between points (0, 73) and (2, 63.9) is $m = \frac{63.9-73}{2-0} = \frac{-9.1}{2} = -4.55$, so the equation of the line is $y = -4.55x + 73$.

Plugging in $x = -5$ gives us $y = -4.55(-5) + 73 = 95.75\ cm$.

Vocabulary

- The **line of best fit** is a good method if the relationship between the dependent and the independent variables is linear. In this section you will learn other methods that are useful even when the relationship isn't linear.

- We use **linear interpolation** to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.

- **For non-linear data,** linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation,** which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.

- **Linear extrapolation** can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Guided Practice

The Center for Disease Control collects information about the health of the American people and behaviors that might lead to bad health. The following table shows the percent of women who smoke during pregnancy.

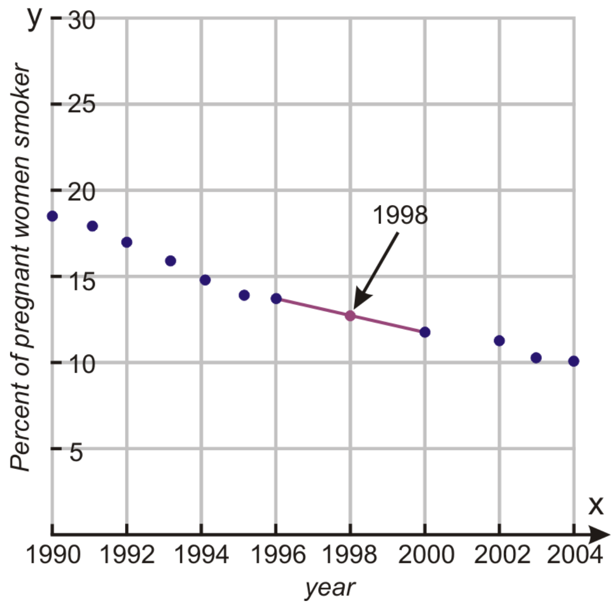
TABLE 4.14:

Year	Percent of pregnant women smokers
1990	18.4
1991	17.7
1992	16.9
1993	15.8
1994	14.6
1995	13.9
1996	13.6
2000	12.2
2002	11.4
2003	10.4
2004	10.2

Estimate the percentage of pregnant women that were smoking in the year 1998.

Solution

We connect the two points on either side of 1998 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$
\begin{aligned}
m &= \frac{12.2-13.6}{2000-1996} = \frac{-1.4}{4} = -0.35 \\
y &= -0.35x + b \\
12.2 &= -0.35(2000) + b \\
b &= 712.2
\end{aligned}
$$

Our equation is $y = -0.35x + 712.2$.

To estimate the percentage of pregnant women who smoked in the year 1998, we plug $x = 1998$ into the equation we just found:

$$y = -0.35(1998) + 712.2 = 12.9\%$$

Practice

1. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1946. Fit a line, by hand, to the data before 1970.
2. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1984. Fit a line, by hand, to the data from 1970 on in order to estimate this accurately.
3. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for males in 1995. Use linear interpolation between the 1990 and 2000 data points.
4. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 1997. Use linear interpolation between the 1996 and 2000 data points.
5. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 2006. Use linear extrapolation with the final two data points.
6. Use the data from Example 3 (*Winning times*) to estimate the winning time for the female 100-meter race in 1920. Use linear extrapolation because the first two or three data points have a different slope than the rest of the data.
7. The table below shows the highest temperature vs. the hours of daylight for the 15^{th} day of each month in the year 2006 in San Diego, California.

TABLE 4.15:

Hours of daylight	High temperature (F)
10.25	60
11.0	62
12	62
13	66
13.8	68
14.3	73
14	86
13.4	75
12.4	71
11.4	66
10.5	73
10	61

(a) What would be a better way to organize this table if you want to make the relationship between daylight hours and temperature easier to see?

(b) Estimate the high temperature for a day with 13.2 hours of daylight using linear interpolation.

(c) Estimate the high temperature for a day with 9 hours of daylight using linear extrapolation. Is the prediction accurate?

(d) Estimate the high temperature for a day with 9 hours of daylight using a line of best fit.

The table below lists expected life expectancies based on year of birth (US Census Bureau). Use it to answer questions 8-15.

TABLE 4.16:

Birth year	Life expectancy in years
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77

1. Make a scatter plot of the data.
2. Use a line of best fit to estimate the life expectancy of a person born in 1955.
3. Use linear interpolation to estimate the life expectancy of a person born in 1955.
4. Use a line of best fit to estimate the life expectancy of a person born in 1976.
5. Use linear interpolation to estimate the life expectancy of a person born in 1976.
6. Use a line of best fit to estimate the life expectancy of a person born in 2012.
7. Use linear extrapolation to estimate the life expectancy of a person born in 2012.
8. Which method gives better estimates for this data set? Why?

The table below lists the high temperature for the fist day of the month for the year 2006 in San Diego, California (Weather Underground). Use it to answer questions 16-21.

TABLE 4.17:

Month number	Temperature (F)
1	63
2	66
3	61
4	64
5	71
6	78
7	88
8	78
9	81
10	75
11	68
12	69

1. Draw a scatter plot of the data.
2. Use a line of best fit to estimate the temperature in the middle of the 4^{th} month (month 4.5).
3. Use linear interpolation to estimate the temperature in the middle of the 4^{th} month (month 4.5).
4. Use a line of best fit to estimate the temperature for month 13 (January 2007).
5. Use linear extrapolation to estimate the temperature for month 13 (January 2007).
6. Which method gives better estimates for this data set? Why?
7. Name a real-world situation where you might want to make predictions based on available data. Would linear extrapolation/interpolation or the best fit method be better to use in that situation? Why?

4.5 Inequality Expressions

Here you'll learn how to write and graph inequalities in one variable on a number line.

Guidance

Dita has a budget of $350 to spend on a rental car for an upcoming trip, but she wants to spend as little of that money as possible. If the trip will last five days, what range of daily rental rates should she be willing to consider?

Like equations, inequalities show a relationship between two expressions. We solve and graph inequalities in a similar way to equations—but when we solve an inequality, the answer is usually a set of values instead of just one value.

When writing inequalities we use the following symbols:

$>$is greater than

$\geq$ is greater than or equal to

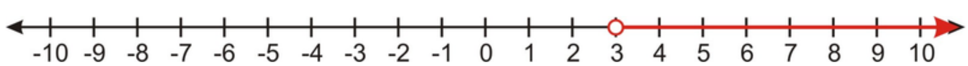
$<$is less than

$\leq$ is less than or equal to

Write and Graph Inequalities in One Variable on a Number Line

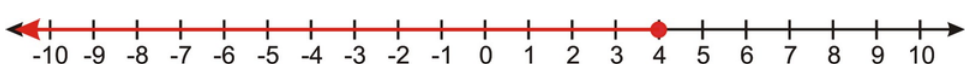
Let's start with the simple inequality $x > 3$.

We read this inequality as "x is greater than 3." The solution is the set of all real numbers that are greater than three. We often represent the solution set of an inequality with a number line graph.



Consider another simple inequality: $x \leq 4$.

We read this inequality as "x is less than or equal to 4." The solution is the set of all real numbers that are equal to four or less than four. We can graph this solution set on the number line.



Notice that we use an empty circle for the endpoint of a strict inequality (like $x > 3$), and a filled circle for one where the equals sign is included (like $x \leq 4$).

Example A

Graph the following inequalities on the number line.

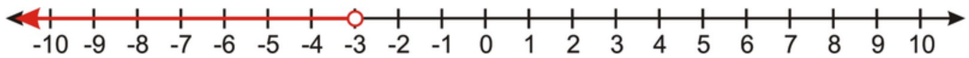
a) $x < -3$

b) $x \geq 6$

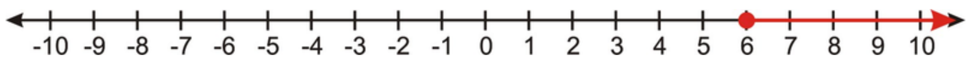
c) $x > 0$

Solution

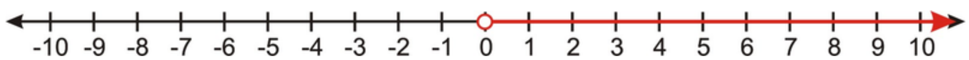
a) The inequality $x < -3$ represents all numbers that are less than -3. The number -3 is not included in the solution, so it is represented by an open circle on the graph.



b) The inequality $x \geq 6$ represents all numbers that are greater than or equal to 6. The number 6 is included in the solution, so it is represented by a closed circle on the graph.



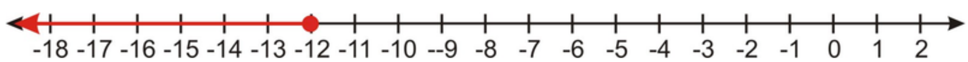
c) The inequality $x > 0$ represents all numbers that are greater than 0. The number 0 is not included in the solution, so it is represented by an open circle on the graph.



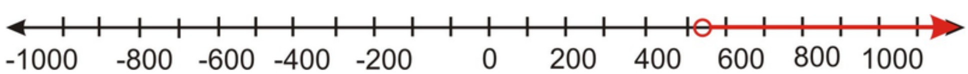
Example B

Write the inequality that is represented by each graph.

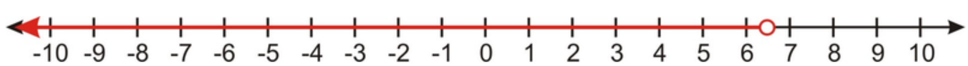
a)



b)



c)



Solution

a) $x \leq -12$

b) $x > 540$

c) $x < 6.5$

Inequalities appear everywhere in real life. Here are some simple examples of real-world applications.

Example C

Write each statement as an inequality and graph it on the number line.

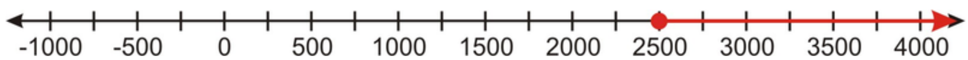
a) You must maintain a balance of at least $2500 in your checking account to get free checking.

b) You must be at least 48 inches tall to ride the "Thunderbolt" Rollercoaster.

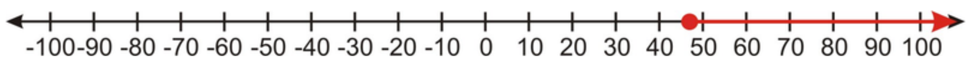
c) You must be younger than 3 years old to get free admission at the San Diego Zoo.

Solution

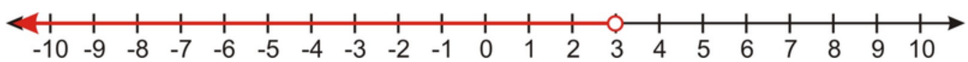
a) The words "at least" imply that the value of \$2500 is included in the solution set, so the inequality is written as $x \geq 2500$.



b) The words "at least" imply that the value of 48 inches is included in the solution set, so the inequality is written as $x \geq 48$.



c) The inequality is written as $x < 3$.



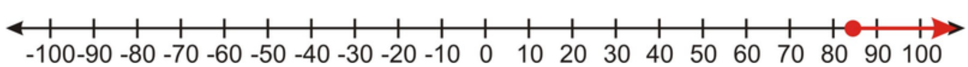
Vocabulary

- The answer to an **inequality** is usually an **interval of values**.

Guided Practice

1. *Graph the inequality* $x \leq 8$ *on the number line.*

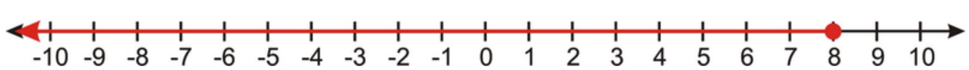
2. *Write the inequality that is represented by the graph below.*



3. *Write the statement, "the speed limit on the interstate is 65 miles per hour or less" as an inequality .*

Solution

1. The inequality $x \leq 8$ represents all numbers that are less than or equal to 8. The number 8 is included in the solution, so it is represented by a closed circle on the graph.

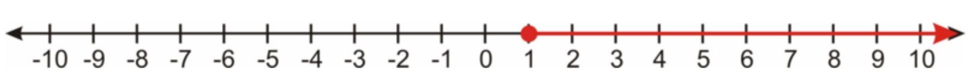


2. $x \geq 85$

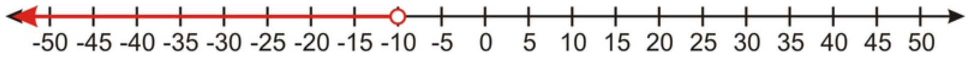
3. Speed limit means the highest allowable speed, so the inequality is written as $x \leq 65$.

Practice

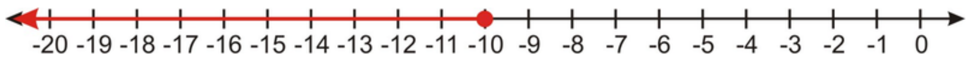
1. Write the inequality represented by the graph.



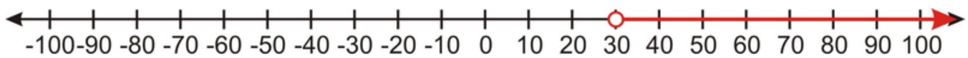
2. Write the inequality represented by the graph.



3. Write the inequality represented by the graph.



4. Write the inequality represented by the graph.



Graph each inequality on the number line.

5. $x < -35$
6. $x > -17$
7. $x \geq 20$
8. $x \leq 3$
9. $x \geq -5$
10. $x > 20$

4.6 Compound Inequalities

Here you will solve two inequalities that have been joined together by the words "and" and "or."

Guidance

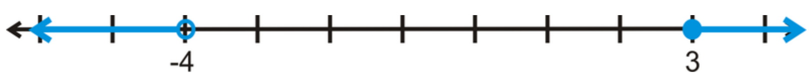
Compound inequalities are inequalities that have been joined by the words "and" or "or." For example:

-2 <x <5 Read, "x is greater than -2 *and* less than 5."

$x \geq 3$ or $x < -4$ Read, "x is greater than or equal to 3 *or* less than -4."

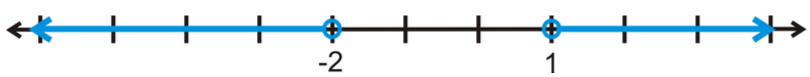
Notice that both of these inequalities have two inequality signs. So, it is like solving or graphing two inequalities at the same time. When graphing, look at the inequality to help you. The first compound inequality above, $-2 < x \leq 5$, has the x in between -2 and 5, so the shading will also be between the two numbers.

And, with the "or" statement, the shading will go in opposite directions.



Example A

Write the inequality statement given by the graph below.



Solution: Because the shading goes in opposite directions, we know this is an "or" statement. Therefore, the statement is $x < -2$ or $x > 1$.

Example B

Solve and graph $-3 < 2x + 5 \leq 11$.

Solution: This is like solving two inequalities at the same time. You can split the statement apart to have two inequalities, $-3 < 2x + 5$ and $2x + 5 \leq 11$ and solve. You can also leave the compound inequality whole to solve.

$$\begin{array}{l} -3 < 2x + \cancel{5} \leq 11 \\ \underline{-5 \qquad\quad -\cancel{5} \quad -5} \\ \frac{-8}{2} < \frac{\cancel{2}x}{\cancel{2}} \leq \frac{6}{2} \\ -4 < x \leq 3 \end{array}$$

Test a solution, $x = 0$:

$$\begin{aligned} -3 < 2(0) + 5 \leq 11 \\ -3 < 5 \leq 11 \end{aligned}$$

Here is the graph:

Example C

Solve and graph $-32 > -5x + 3$ or $x - 4 \leq 2$.

Solution: When solving an "or" inequality, solve the two inequalities separately, but show the solution on the same number line.

$$\begin{array}{lcl} -32 > -5x + \cancel{3} & \text{or} & x - \cancel{4} \leq 2 \\ \underline{-3 \qquad\quad -\cancel{3}} & & \underline{+\cancel{4} \quad +4} \\ \dfrac{-35}{-5} > \dfrac{-\cancel{5}x}{-\cancel{5}} & & x \leq 6 \\ 7 < x & & \end{array}$$

Notice that in the first inequality, we had to flip the inequality sign because we divided by -5. Also, it is a little more complicated to test a solution for these types of inequalities. You still test one point, but it will only work for one of the inequalities. Let's test $x = 10$. First inequality: $-32 > -5(10) + 3 \rightarrow -32 > -47$. Second inequality: $10 - 4 \leq 2 \rightarrow 5 \not\leq 2$. Because $x = 10$ works for the first inequality, is a solution. Here is the graph.

Intro Problem Revisit Writing the grading as an expression, we get $0.4(84) + 0.6x$ where x is the final exam score. Madison wants to get an A, so we will have a compound inequality that ranges between 90 and 100.

$$\begin{aligned} 90 \leq 33.6 + 0.6x \leq 100 \\ 56.4 \leq 0.6x \leq 66.4 \\ 94 \leq x \leq 110.67 \end{aligned}$$

Unless Mr. Garcia offers extra credit, Madison can't score higher than 100. So, she has to score at least 94 or more, up to 100, to get an A.

Guided Practice

1. Graph $-7 \leq x \leq -1$ on a number line.

Solve the following compound inequalities and graph.

2. $5 \leq -\frac{2}{3}x + 1 \leq 15$

3. $\frac{x}{4} - 7 > 5$ or $\frac{8}{5}x + 2 \leq 18$

Answers

1. This is an "and" inequality, so the shading will be between the two numbers.

2. Solve this just like Example B.

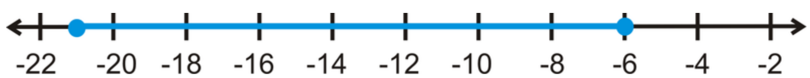
$$\begin{array}{c} 5 \leq -\frac{2}{3}x + \cancel{1} \leq 15 \\ \underline{-1 \qquad\quad -\cancel{1} \quad -1} \\ 4 \leq -\frac{2}{3}x \leq 14 \\ -\frac{3}{2}\left(4 \leq -\frac{2}{3}x \leq 14\right) \\ -6 \geq x \geq -21 \end{array}$$

Test a solution, $x = -10$:

$$\begin{array}{r} 5 \leq -\frac{2}{3}(-12) + 1 \leq 15 \\ 5 \leq 9 \leq 15 \end{array}$$

This solution can also be written $-21 \leq x \leq -6$.

The graph is:



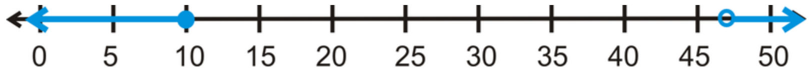
3. This is an "or" compound inequality. Solve the two inequalities separately.

$$\begin{array}{ccc} \frac{x}{4} - \cancel{7} > 5 & or & \frac{8}{5}x + \cancel{2} \leq 18 \\ \underline{+\cancel{7} \ +7} & & \underline{-\cancel{2} \ -2} \\ \cancel{4} \cdot \frac{x}{4} > 12 \cdot 4 & or & \cancel{\frac{5}{8}} \cdot \cancel{\frac{8}{5}}x \leq 16 \cdot \frac{5}{8} \\ x > 48 & or & x \leq 10 \end{array}$$

Test a solution, $x = 0$:

$$\begin{array}{ccc} \frac{0}{4} - 7 > 5 & or & \frac{8}{5}(0) + 2 \leq 18 \\ -7 \not> 5 & or & 2 \leq 18 \end{array}$$

Notice that $x = 0$ is a solution for the second inequality, which makes it a solution for the entire compound inequality. Here is the graph:



On problems 2 and 3 we changed the scale of the number line to accommodate the solution.

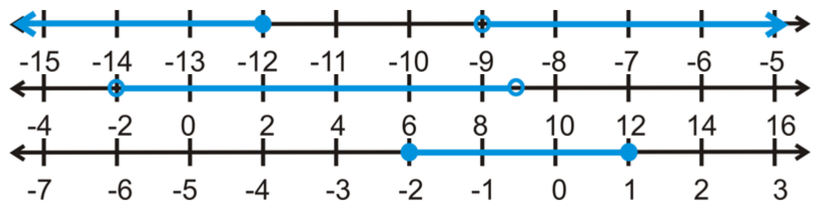
Practice

Graph the following compound inequalities. Use an appropriate scale.

1. $-1 < x < 8$
2. $x > 5$ or $x \leq 3$
3. $-4 \leq x \leq 0$

Write the compound inequality that best fits each graph below.

4.
5.
6.



Solve each compound inequality and graph the solution.

7. $-11 < x - 9 \leq 2$
8. $8 \leq 3 - 5x < 28$
9. $2x - 7 > -13$ or $\frac{1}{3}x + 5 \leq 1$
10. $0 < \frac{x}{5} < 4$
11. $-4x + 9 < 35$ or $3x - 7 \leq -16$
12. $\frac{3}{4}x + 7 \geq -29$ or $16 - x > 2$
13. $3 \leq 6x - 15 < 51$
14. $-20 < -\frac{3}{2}x + 1 < 16$
15. **Challenge** Write a compound inequality whose solutions are all real numbers. Show why this is true.

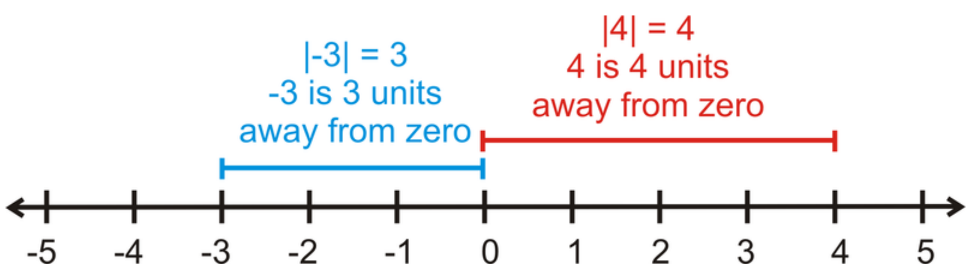
4.7 Solving Absolute Value Equations

Here you'll learn to solve absolute value equations.

To determine the height of skeletal remains, archaeologists use the equation $H = 2.26f + 66.4$, where H is the height in centimeters and f is the length of the skeleton's femur (also in cm). The equation has a margin of error of $\pm 3.42 cm$. Dr. Jordan found a skeletal femur that is 46.8 cm. Determine the greatest height and the least height of this person.

Guidance

Absolute value is the distance a number is from zero. Because distance is always positive, the absolute value will always be positive. Absolute value is denoted with two vertical lines around a number, $|x|$.



$|5| = 5$ $\quad |-9| = 9$ $\quad |0| = 0$ $\quad |-1| = 1$

When solving an absolute value equation, the value of x could be two different possibilities; whatever makes the absolute value positive OR whatever makes it negative. Therefore, there will always be TWO answers for an absolute value equation.

If $|x| = 1$, then x can be 1 or -1 because $|1| = 1$ and $|-1| = 1$.

If $|x| = 15$, then x can be 15 or -15 because $|15| = 15$ and $|-15| = 15$.

From these statements we can conclude:

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Example A

Determine if $x = -12$ is a solution to $|2x - 5| = 29$.

Solution: Plug in -12 for x to see if it works.

$$\begin{aligned} |2(-12) - 5| &= 29 \\ |-24 - 5| &= 29 \\ |-29| &= 29 \end{aligned}$$

-12 is a solution to this absolute value equation.

Example B

Solve $|x+4| = 11$.

Solution: There are going to be two answers for this equation. $x+4$ can equal 11 or -11.

$$\begin{array}{ccc} & |x+4| = 11 & \\ & \swarrow \searrow & \\ x+4 = 11 & & x+4 = -11 \\ & or & \\ x = 7 & & x = -15 \end{array}$$

Test the solutions:

$$\begin{array}{cc} |7+4| = 11 & |-15+4| = 11 \\ |11| = 11 & |-11| = 11 \end{array}$$

Example C

Solve $\left|\frac{2}{3}x-5\right| = 17$.

Solution: Here, what is inside the absolute value can be equal to 17 or -17.

$$\begin{array}{ccc} & \left|\frac{2}{3}x-5\right| = 17 & \\ & \swarrow \searrow & \\ \frac{2}{3}x-5 = 17 & & \frac{2}{3}x-5 = -17 \\ \frac{2}{3}x = 22 & or & \frac{2}{3}x = -12 \\ x = 22 \cdot \frac{3}{2} & & x = -12 \cdot \frac{3}{2} \\ x = 33 & & x = -18 \end{array}$$

Test the solutions:

$$\begin{array}{cc} \left|\frac{2}{3}(33)-5\right| = 17 & \left|\frac{2}{3}(-18)-5\right| = 17 \\ |22-5| = 17 & |-12-5| = 17 \\ |17| = 17 & |-17| = 17 \end{array}$$

Intro Problem Revisit First, we need to find the height of the skeleton using the equation $H = 2.26f + 66.4$, where $f = 46.8$.

$$H = 2.26(46.8) + 66.4$$
$$H = 172.168cm$$

Now, let's use an absolute value equation to determine the margin of error, and thus the greatest and least heights.

$$|x - 172.168| = 3.42$$
$$\swarrow \searrow$$
$$x - 172.168 = 3.42 \quad x - 172.168 = -3.42$$
$$or$$
$$x = 175.588 \qquad x = 168.748$$

So the person could have been a maximum of 175.588 cm or a minimum of 168.748 cm. In inches, this would be 69.13 and 66.44 inches, respectively.

Guided Practice

1. Is $x = -5$ a solution to $|3x + 22| = 6$?

Solve the following absolute value equations.

2. $|6x - 11| + 2 = 41$

3. $\left|\frac{1}{2}x + 3\right| = 9$

Answers

1. Plug in -5 for x to see if it works.

$$|3(-5) + 22| = 6$$
$$|-15 + 22| = 6$$
$$|-7| \neq 6$$

-5 is not a solution because $|-7| = 7$, not 6.

2. Find the two solutions. Because there is a 2 being added to the left-side of the equation, we first need to subtract it from both sides so the absolute value is by itself.

$$|6x - 11| + 2 = 41$$
$$|6x - 11| = 39$$
$$\swarrow \searrow$$
$$6x - 11 = 39 \quad 6x - 11 = -39$$
$$6x = 50 \qquad 6x = -28$$
$$x = \frac{50}{6} \quad or \quad x = -\frac{28}{6}$$
$$= \frac{25}{3} \; or \; 8\frac{1}{3} \qquad = -\frac{14}{3} \; or \; -4\frac{2}{3}$$

Check both solutions. It is easier to check solutions when they are improper fractions.

$$\begin{aligned} \left|6\left(\frac{25}{3}\right)-11\right| &= 39 & \left|6\left(-\frac{14}{3}\right)-11\right| &= 39 \\ |50-11| &= 39 \quad \textit{and} & |-28-11| &= 39 \\ |39| &= 39 & |-39| &= 39 \end{aligned}$$

3. What is inside the absolute value is equal to 9 or -9.

$$\left|\frac{1}{2}x+3\right| = 9$$

$$\swarrow \searrow$$

$$\begin{aligned} \frac{1}{2}x+3 &= 9 & \frac{1}{2}x+3 &= -9 \\ \frac{1}{2}x &= 6 \quad \textit{or} & \frac{1}{2}x &= -12 \\ x &= 12 & x &= -24 \end{aligned}$$

Test solutions:

$$\begin{aligned} \left|\frac{1}{2}(12)+3\right| &= 9 & \left|\frac{1}{2}(-24)+3\right| &= 9 \\ |6+3| &= 9 & |-12+3| &= 9 \\ |9| &= 9 & |-9| &= 9 \end{aligned}$$

Vocabulary

Absolute Value

The positive distance from zero a given number is.

Practice

Determine if the following numbers are solutions to the given absolute value equations.

1. $|x-7| = 16; 9$
2. $|\frac{1}{4}x+1| = 4; -8$
3. $|5x-2| = 7; -1$

Solve the following absolute value equations.

4. $|x+3| = 8$
5. $|2x| = 9$
6. $|2x+15| = 3$
7. $\left|\frac{1}{3}x-5\right| = 2$
8. $\left|\frac{x}{6}+4\right| = 5$

9. $|7x-12|=23$
10. $\left|\frac{3}{5}x+2\right|=11$
11. $|4x-15|+1=18$
12. $|-3x+20|=35$
13. $|12x-18|=0$
14. What happened in #13? Why do you think that is?
15. **Challenge** When would an absolute value equation have no solution? Give an example.

4.8 Solving Absolute Value Inequalities

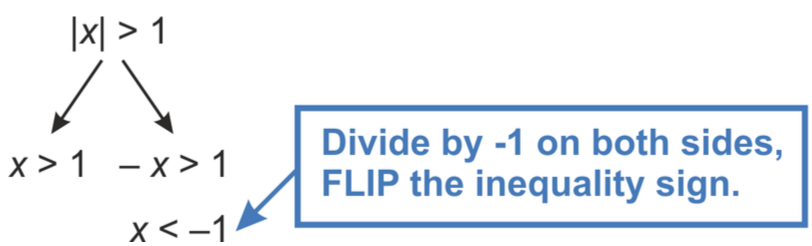
Here you'll learn how to solve absolute value inequalities.

The tolerance for the weight of a volleyball is 2.6 grams. If the average volleyball weighs 260 grams, what is the range of weights for a volleyball?

Guidance

Like absolute value equations, absolute value inequalities also will have two answers. However, they will have a range of answers, just like compound inequalities.

$|x| > 1$ This inequality will have two answers, when x is 1 and when $-x$ is 1. But, what about the inequality sign? The two possibilities would be:



Notice in the second inequality, we did not write $x > -1$. This is because what is inside the absolute value sign can be positive or negative. Therefore, if x is negative, then $-x > 1$. It is a very important difference between the two inequalities. Therefore, for the first solution, we leave the inequality sign the same and for the second solution we need to change the sign of the answer AND flip the inequality sign.

Example A

Solve $|x+2| \leq 10$.

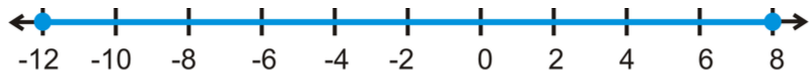
Solution: There will be two solutions, one with the answer and sign unchanged and the other with the inequality sign flipped and the answer with the opposite sign.

$$\begin{array}{cc} \multicolumn{2}{c}{|x+2| \leq 10} \\ \swarrow & \searrow \\ x+2 \leq 10 & x+2 \geq -10 \\ x \leq 8 & x \geq -12 \end{array}$$

Test a solution, $x = 0$:

$$\begin{aligned} |0+2| &\leq 10 \\ |2| &\leq 10 \end{aligned}$$

When graphing this inequality, we have



Notice that this particular absolute value inequality has a solution that is an "and" inequality because the solution is between two numbers.

If $|ax+b| < c$ where $a > 0$ and $c > 0$, then $-c < ax+b < c$.

If $|ax+b| \leq c$ where $a > 0$ and $c > 0$, then $-c \leq ax+b \leq c$.

If $|ax+b| > c$ where $a > 0$ and $c > 0$, then $ax+b < -c$ or $ax+b > c$.

If $|ax+b| \geq c$ where $a > 0$ and $c > 0$, then $ax+b \leq -c$ or $ax+b \geq c$.

If $a < 0$, we will have to divide by a negative and have to flip the inequality sign. This would change the end result. If you are ever confused by the rules above, always test one or two solutions and graph it.

Example B

Solve and graph $|4x-3| > 9$.

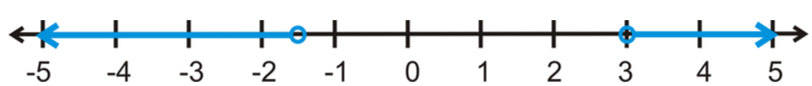
Solution: Break apart the absolute value inequality to find the two solutions.

$$\begin{array}{cc} \multicolumn{2}{c}{|4x-3| > 9} \\ \swarrow & \searrow \\ 4x-3 > 9 & 4x-3 < -9 \\ 4x > 12 & 4x < -6 \\ x > 3 & x < -\frac{3}{2} \end{array}$$

Test a solution, $x = 5$:

$$\begin{aligned} |4(5)-3| &> 9 \\ |20-3| &> 9 \\ 17 &> 9 \end{aligned}$$

The graph is:



Example C

Solve $|-2x+5| < 11$.

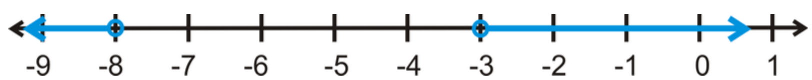
Solution: In this example, the rules above do not apply because $a < 0$. At first glance, this should become an "and" inequality. But, because we will have to divide by a negative number, a, the answer will be in the form of an "or" compound inequality. We can still solve it the same way we have solved the other examples.

$$|-2x+5| < 11$$

$$\begin{aligned} -2x+5 &< 11 & -2x+5 &> -11 \\ -2x &< 6 & -2x &> -16 \\ x &> -3 & x &< -8 \end{aligned}$$

The solution is less than -8 or greater than -3.

The graph is:



When $a < 0$ for an absolute value inequality, it switches the results of the rules listed above.

Intro Problem Revisit Set up an absolute value inequality. w is the range of weights of the volleyball.

$$|w-260| \le 2.6$$

$$\begin{aligned} w-260 &\le 2.6 & w-260 &\ge -2.6 \\ w &\le 262.6 & w &\ge 257.4 \end{aligned}$$

So, the range of the weight of a volleyball is $257.4 \le w \le 262.6$ grams.

Guided Practice

1. Is $x = -4$ a solution to $|15-2x| > 9$?

2. Solve and graph $\left|\frac{2}{3}x+5\right| \le 17$.

Answers

1. Plug in -4 for x to see if it works.

$$\begin{aligned} |15-2(-4)| &> 9 \\ |15+8| &> 9 \\ |23| &> 9 \\ 23 &> 9 \end{aligned}$$

Yes, -4 works, so it is a solution to this absolute value inequality.

2. Split apart the inequality to find the two answers.

$$\left|\frac{2}{3}x+5\right| \le 17$$

$$\begin{array}{cc} \left|\frac{2}{3}x+5\right| \le 17 & \frac{2}{3}x+5 \ge -17 \\ \frac{2}{3}x \le 12 & \frac{2}{3}x \ge -22 \\ x \le 12 \cdot \frac{3}{2} & x \ge -22 \cdot \frac{3}{2} \\ x \le 18 & x \ge -33 \end{array}$$

Test a solution, $x = 0$:

$$\begin{aligned} \left|\frac{2}{3}(0)+5\right| &\le 17 \\ |5| &\le 17 \\ 5 &\le 17 \end{aligned}$$

Practice

Determine if the following numbers are solutions to the given absolute value inequalities.

1. $|x-9| > 4; 10$
2. $\left|\frac{1}{2}x-5\right| \le 1; 8$
3. $|5x+14| \ge 29; -8$

Solve and graph the following absolute value inequalities.

4. $|x+6| > 12$
5. $|9-x| \le 16$
6. $|2x-7| \ge 3$
7. $|8x-5| < 27$
8. $\left|\frac{5}{6}x+1\right| > 6$
9. $|18-4x| \le 2$
10. $\left|\frac{3}{4}x-8\right| > 13$
11. $|6-7x| \le 34$
12. $|19+3x| \ge 46$

Solve the following absolute value inequalities. a is greater than zero.

13. $|x-a| > a$
14. $|x+a| \le a$
15. $|a-x| \le a$

4.9 Linear Inequalities in Two Variables

Here you'll learn how to graph linear inequalities in two variables of the form $y > mx+b$ or $y < mx+b$. You'll also solve real-world problems involving such inequalities.

Guidance

The general procedure for graphing inequalities in two variables is as follows:

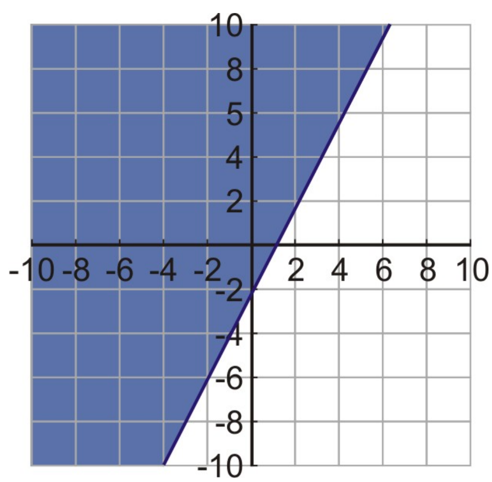
1. Re-write the inequality in slope-intercept form: $y = mx+b$. Writing the inequality in this form lets you know the direction of the inequality.
2. Graph the line of the equation $y = mx+b$ using your favorite method (plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
3. Shade the half plane above the line if the inequality is "greater than." Shade the half plane under the line if the inequality is "less than."

Example A

Graph the inequality $y \geq 2x-3$.

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x-3$; then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.



Example B

Graph the inequality $5x-2y > 4$.

Solution

First we need to rewrite the inequality in slope-intercept form:

$$-2y > -5x + 4$$
$$y < \frac{5}{2}x - 2$$

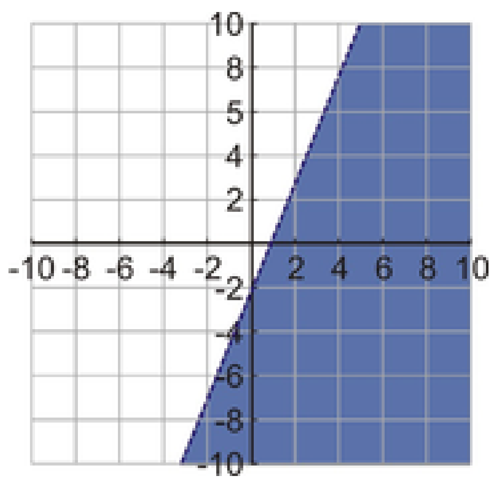
Notice that the inequality sign changed direction because we divided by a negative number.

To graph the equation, we can make a table of values:

TABLE 4.18:

x	y
-2	$\frac{5}{2}(-2) - 2 = -7$
0	$\frac{5}{2}(0) - 2 = -2$
2	$\frac{5}{2}(2) - 2 = 3$

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example C

A retailer sells two types of coffee beans. One type costs \$9 per pound and the other type costs \$7 per pound. Find all the possible amounts of the two different coffee beans that can be mixed together to get a quantity of coffee beans costing \$8.50 or less.

Solution

Let $x =$ weight of \$9 per pound coffee beans in pounds.

Let $y =$ weight of \$7 per pound coffee beans in pounds.

The cost of a pound of coffee blend is given by $9x + 7y$.

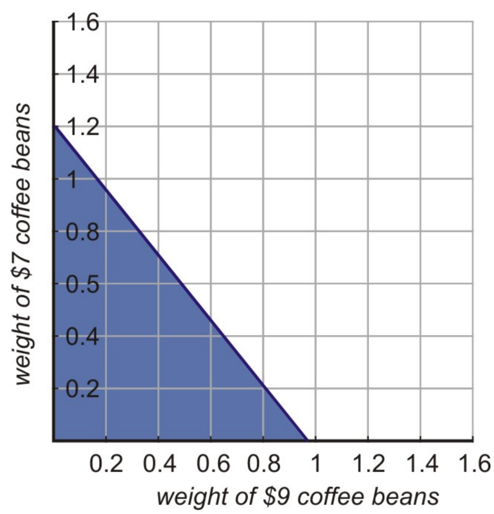
We are looking for the mixtures that cost \$8.50 or less. We write the inequality $9x + 7y \leq 8.50$.

Since this inequality is in standard form, it's easiest to graph it by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $7y = 8.50$ or $y = \frac{8.50}{7} \approx 1.21$. When $y = 0$, we have $9x = 8.50$ or $x = \frac{8.50}{9} \approx 0.94$. We can then graph the line that includes those two points.

Now we have to figure out which side of the line to shade. In $y-$intercept form, we shade the area **below** the line when the inequality is "less than." But in standard form that's not always true. We could convert the inequality to $y-$intercept form to find out which side to shade, but there is another way that can be easier.

The other method, which works for any linear inequality in any form, is to plug a random point into the inequality and see if it makes the inequality true. Any point that's not on the line will do; the point (0, 0) is usually the most convenient.

In this case, plugging in 0 for x and y would give us $9(0) + 7(0) \leq 8.50$, which is true. That means we should shade the half of the plane that includes (0, 0). If plugging in (0, 0) gave us a false inequality, that would mean that the solution set is the part of the plane that does *not* contain (0, 0).



Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always nonnegative, so points outside the first quadrant don't represent real-world solutions to this problem.

Vocabulary

- For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

- The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

>The solution set is the half plane above the line.

$\geq$ The solution set is the half plane above the line and also all the points on the line.

<The solution set is the half plane below the line.

$\leq$ The solution set is the half plane below the line and also all the points on the line.

Guided Practice

Julius has a job as an appliance salesman. He earns a commission of \$60 for each washing machine he sells and \$130 for each refrigerator he sells. How many washing machines and refrigerators must Julius sell in order to make \$1000 or more in commissions?

Solution

Let $x =$ number of washing machines Julius sells.

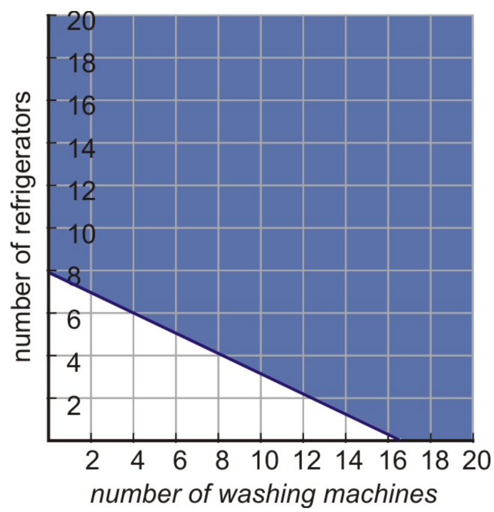
Let $y =$ number of refrigerators Julius sells.

The total commission is $60x + 130y$.

We're looking for a total commission of \$1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Once again, we can do this most easily by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $130y = 1000$, or $y = \frac{1000}{30} \approx 7.69$. When $y = 0$, we have $60x = 1000$, or $x = \frac{1000}{60} \approx 16.67$.

We draw a solid line connecting those points, and shade above the line because the inequality is "greater than." We can check this by plugging in the point (0, 0): selling 0 washing machines and 0 refrigerators would give Julius a commission of \$0, which is *not* greater than or equal to \$1000, so the point (0, 0) is *not* part of the solution; instead, we want to shade the side of the line that does *not* include it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius's commission should be non-negative.

Practice

Graph the following inequalities on the coordinate plane.

1. $y \leq 4x + 3$
2. $y > -\frac{x}{2} - 6$
3. $3x - 4y \geq 12$
4. $x + 7y < 5$
5. $6x + 5y > 1$
6. $y + 5 \leq -4x + 10$
7. $x - \frac{1}{2}y \geq 5$
8. $6x + y < 20$
9. $30x + 5y < 100$

10. Remember what you learned in the last chapter about families of lines.
    a. What do the graphs of $y > x+2$ and $y < x+5$ have in common?
    b. What do you think the graph of $x+2 < y < x+5$ would look like?
11. How would the answer to problem 6 change if you subtracted 2 from the right-hand side of the inequality?
12. How would the answer to problem 7 change if you added 12 to the right-hand side?
13. How would the answer to problem 8 change if you flipped the inequality sign?
14. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and nighttime minutes could you use in one week if you wanted to pay less than $20?
15. Suppose you are graphing the inequality $y > 5x$.
    a. Why can't you plug in the point (0, 0) to tell you which side of the line to shade?
    b. What happens if you do plug it in?
    c. Try plugging in the point (0, 1) instead. Now which side of the line should you shade?
16. A theater wants to take in at least $2000 for a certain matinee. Children's tickets cost $5 each and adult tickets cost $10 each.
    a. If x represents the number of adult tickets sold and y represents the number of children's tickets, write an inequality describing the number of tickets that will allow the theater to meet their minimum take.
    b. If 100 children's tickets and 100 adult tickets have already been sold, what inequality describes how many *more* tickets of both types the theater needs to sell?
    c. If the theater has only 300 seats (so only 100 are still available), what inequality describes the *maximum* number of additional tickets of both types the theater can sell?

CHAPTER 5 Chapter 5: Systems

Chapter Outline

5.1 Solving Systems with One Solution Using Graphing

Here you'll learn how to graph lines to identify the unique solution to a system of linear equations.

Guidance

In this lesson we will be using various techniques to graph the pairs of lines in systems of linear equations to identify the point of intersection or the solution to the system. It is important to use graph paper and a straightedge to graph the lines accurately. Also, you are encouraged to check your answer algebraically as described in the previous lesson.

Example A

Graph and solve the system:

$$\begin{aligned} y &= -x+1 \\ y &= \frac{1}{2}x-2 \end{aligned}$$

Solution:

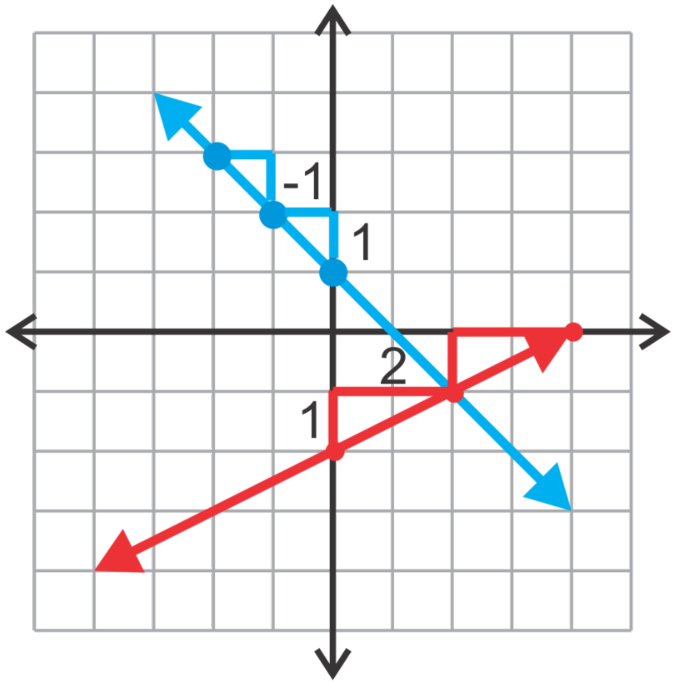
Since both of these equations are written in slope intercept form, we can graph them easily by plotting the $y-$intercept point and using the slope to locate additional points on each line.

The equation, $y = -x+1$, graphed in **blue**, has $y-$intercept 1 and slope $-\frac{1}{1}$.

The equation, $y = \frac{1}{2}x-2$, graphed in **red**, has $y-$intercept -2 and slope $\frac{1}{2}$.

Now that both lines have been graphed, the intersection is observed to be the point (2, -1).

Check this solution algebraically by substituting the point into both equations.



Equation 1: $y = -x + 1$, making the substitution gives: $(-1) = (-2) + 1$.

Equation 2: $y = \frac{1}{2}x - 2$, making the substitution gives: $-1 = \frac{1}{2}(2) - 2$.

(2, -1) is the solution to the system.

Example B

Graph and solve the system:

$$\begin{aligned} 3x + 2y &= 6 \\ y &= -\frac{1}{2}x - 1 \end{aligned}$$

Solution: This example is very similar to the first example. The only difference is that equation 1 is not in slope intercept form. We can either solve for y to put it in slope intercept form or we can use the intercepts to graph the equation. To review using intercepts to graph lines, we will use the latter method.

Recall that the $x-$intercept can be found by replacing y with zero and solving for x:

$$\begin{aligned} 3x + 2(0) &= 6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Similarly, the $y-$intercept is found by replacing x with zero and solving for y:

$$\begin{aligned} 3(0) + 2y &= 6 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

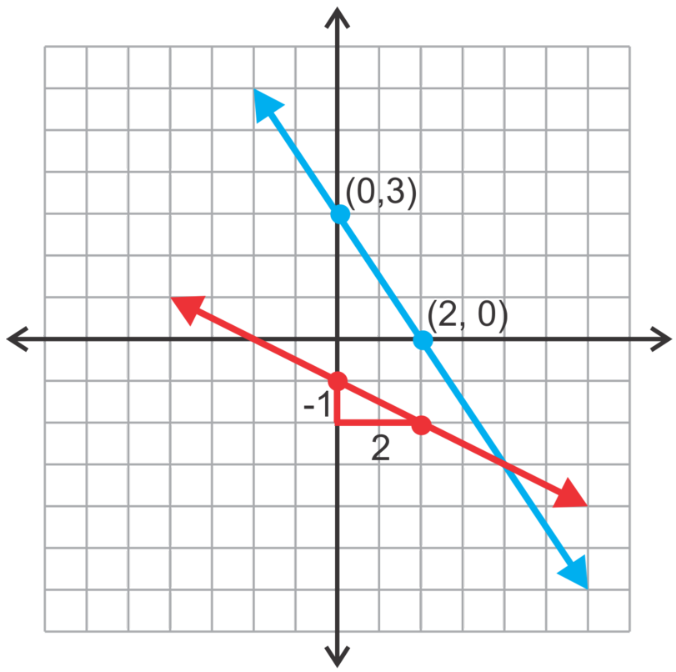
We have two points, (2, 0) and (0, 3) to plot and graph this line. Equation 2 can be graphed using the $y-$intercept and slope as shown in Example A.

Now that both lines are graphed we observe that their intersection is the point (4, -3).

Finally, check this solution by substituting it into each of the two equations.

Equation 1: $3x+2y=6; 3(4)+2(-3)=12-6=6$

Equation 2: $y=-\frac{1}{2}x-1; -3=-\frac{1}{2}(4)-1$



Example C

In this example we will use technology to solve the system:

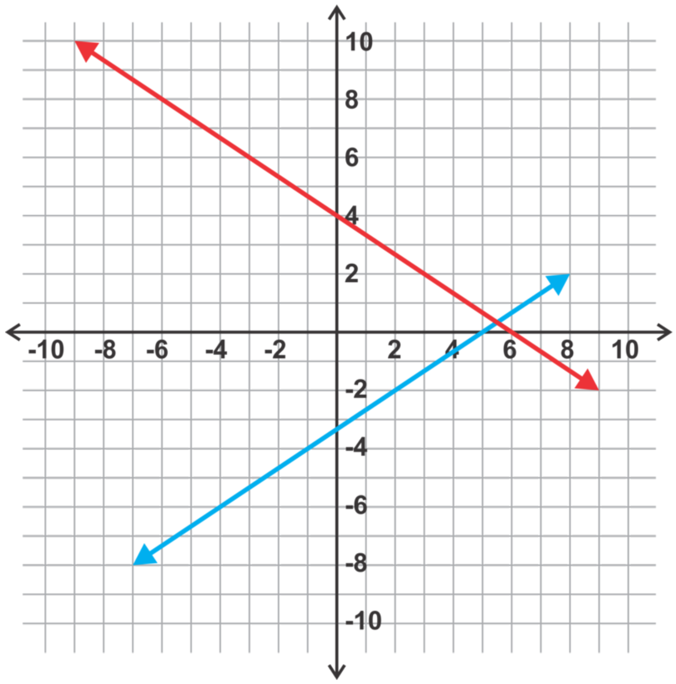
$$\begin{aligned} 2x-3y &= 10 \\ y &= -\frac{2}{3}x+4 \end{aligned}$$

This process may vary somewhat based on the technology you use. All directions here can be applied to the TI-83 or 84 (plus, silver, etc) calculators.

Solution: The first step is to graph these equations on the calculator. The first equation must be rearranged into slope intercept form to put in the calculator.

$$\begin{aligned} 2x-3y &= 10 \\ -3y &= -2x+10 \\ y &= \frac{-2x+10}{-3} \\ y &= \frac{2}{3}x-\frac{10}{3} \end{aligned}$$

The graph obtained using the calculator should look like this:



The first equation, $y = \frac{2}{3}x - \frac{10}{3}$, is graphed in **blue**. The second equation, $y = -\frac{2}{3}x + 4$, is graphed in **red**.

The solution does not lie on the "grid" and is therefore difficult to observe visually. With technology we can calculate the intersection. If you have a TI-83 or 84, use the CALC menu, select INTERSECT. Then select each line by pressing ENTER on each one. The calculator will give you a "guess." Press ENTER one more time and the calculator will then calculate the intersection of (5.5, .333...). We can also write this point as $\left(\frac{11}{2}, \frac{1}{3}\right)$. Check the solution algebraically.

Equation 1: $2x - 3y = 10; 2\left(\frac{11}{2}\right) - 3\left(\frac{1}{3}\right) = 11 - 1 = 10$

Equation 2: $y = -\frac{2}{3}x + 4; -\frac{2}{3}\left(\frac{11}{2}\right) + 4 = -\frac{11}{3} + \frac{12}{3} = \frac{1}{3}$

If you do not have a TI-83 or 84, the commands might be different. Check with your teacher.

Intro Problem Revisit The system of linear equations represented by this situation is:

$$\begin{aligned} 2c1 + 4c2 &= 70 \\ c1 + 5c2 &= 50 \end{aligned}$$

If you plot both of these linear equations on the same graph, you find that the point of intersection is (25, 5). Therefore coin one has a value of 25 cents and coin two has a value of 5 cents.

Guided Practice

Solve the following systems by graphing. Use technology for problem 3.

1.

$$\begin{aligned} y &= 3x - 4 \\ y &= 2 \end{aligned}$$

2.

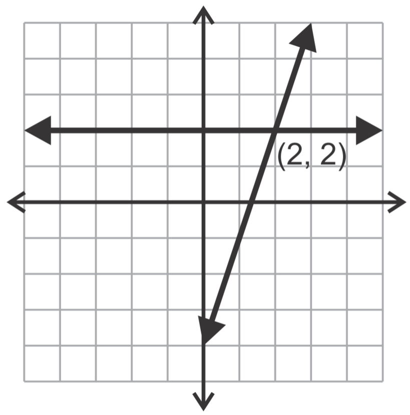
$$2x - y = -4$$
$$2x + 3y = -12$$

3.

$$5x + y = 10$$
$$y = \frac{2}{3}x - 7$$

Answers

1.



The first line is in slope intercept form and can be graphed accordingly.

The second line is a horizontal line through (0, 2).

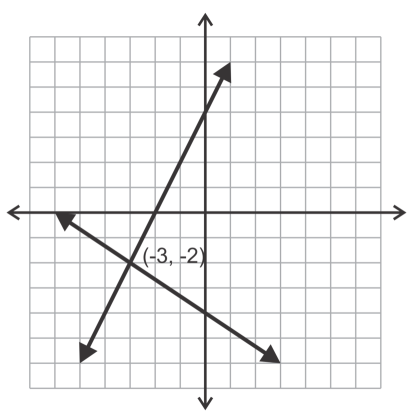
The graph of the two equations is shown below. From this graph the solution appears to be (2, 2).

Checking this solution in each equation verifies that it is indeed correct.

Equation 1: $2 = 3(2) - 4$

Equation 2: $2 = 2$

2.



Neither of these equations is in slope intercept form. The easiest way to graph them is to find their intercepts as shown in Example B.

Equation 1: Let $y = 0$ to find the $x-$intercept.

$$\begin{aligned} 2x - y &= -4 \\ 2x - 0 &= -4 \\ x &= -2 \end{aligned}$$

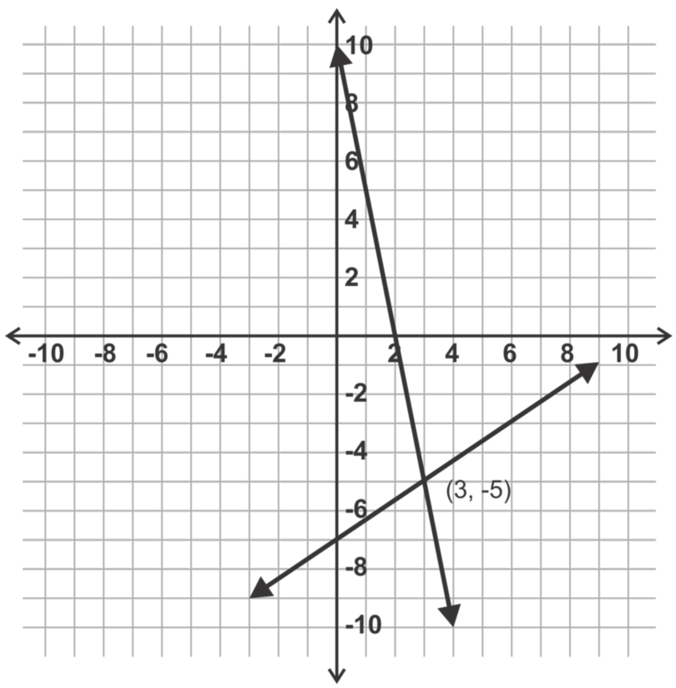
Now let $x = 0$, to find the $y-$intercept.

$$\begin{aligned} 2x - y &= -4 \\ 2(0) - y &= -4 \\ y &= 4 \end{aligned}$$

Now we can use (-2, 0) and (0, 4) to graph the line as shown in the diagram. Using the same process, the intercepts for the second line can be found to be (-6, 0) and (0, -4).

Now the solution to the system can be observed to be (-3, -2). This solution can be verified algebraically as shown in the first problem.

3.



The first equation here must be rearranged to be $y = -5x + 10$ before it can be entered into the calculator. The second equation can be entered as is.

Using the calculate menu on the calculator the solution is (3, -5).

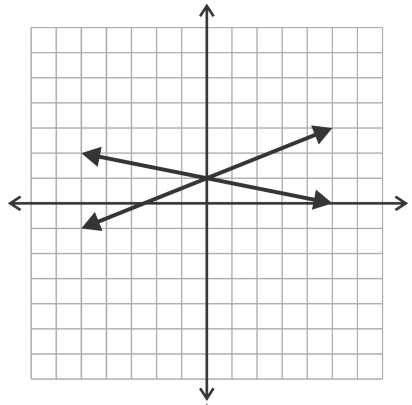
Remember to verify this solution algebraically as a way to check your work.

Practice

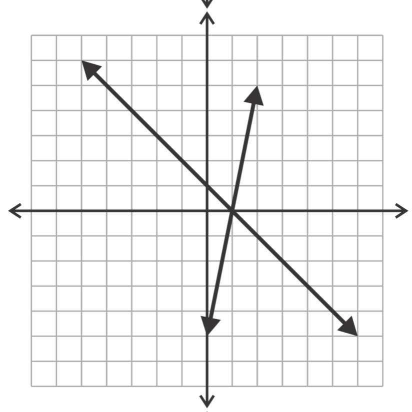
Match the system of linear equations to its graph and state the solution.

1.

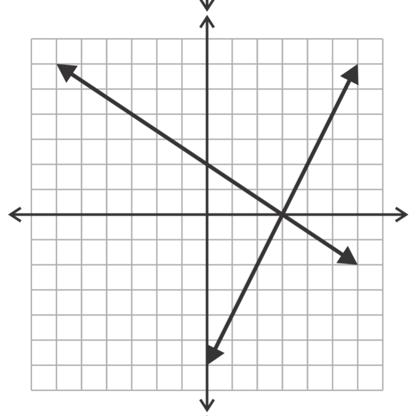
$$3x+2y=-2$$
$$x-y=-4$$



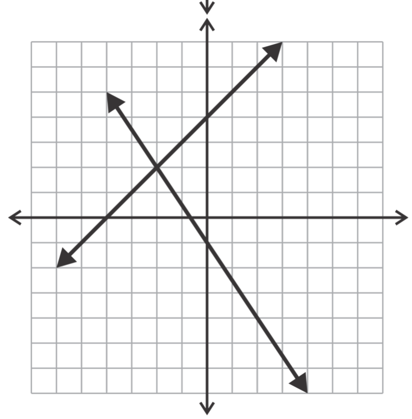
a.



b.



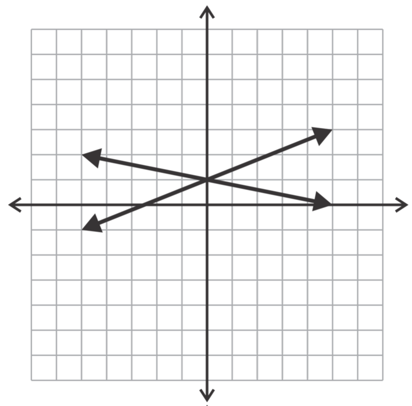
c.



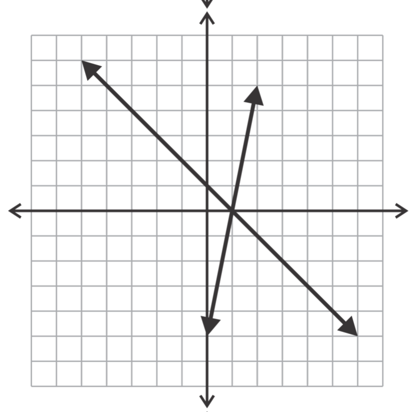
d.

2.

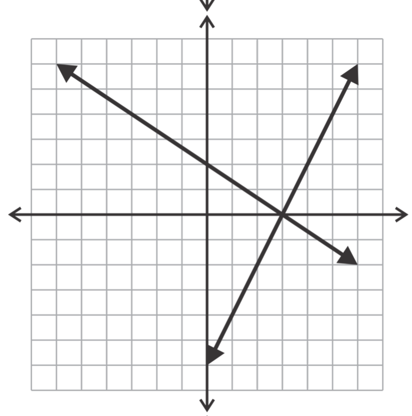
$$2x - y = 6$$
$$2x + 3y = 6$$



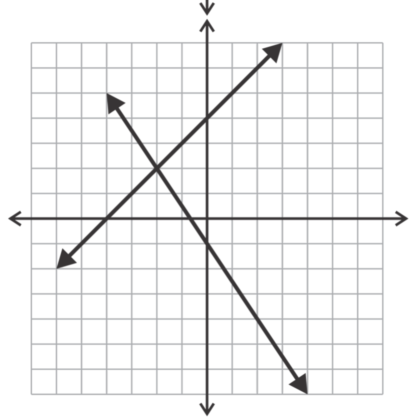
a.



b.



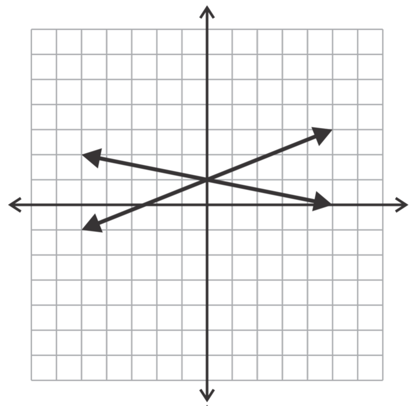
c.

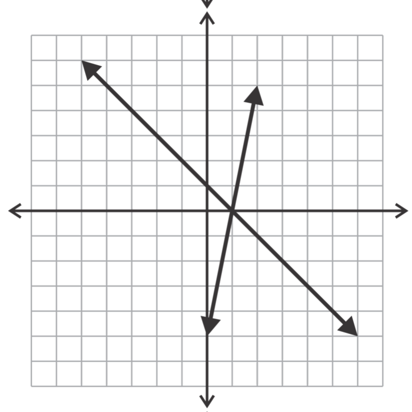


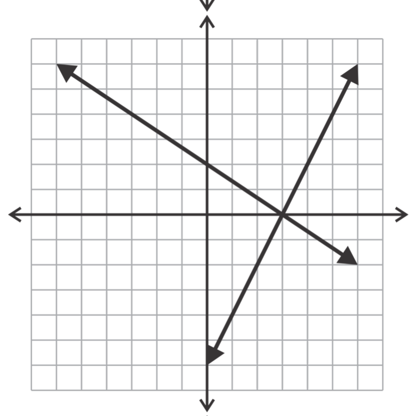
d.

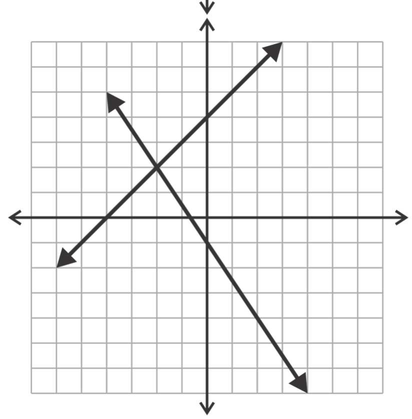
3.

$$2x-5y=-5$$
$$x+5y=5$$


a.

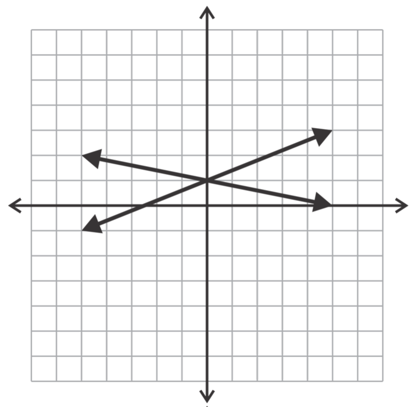

b.

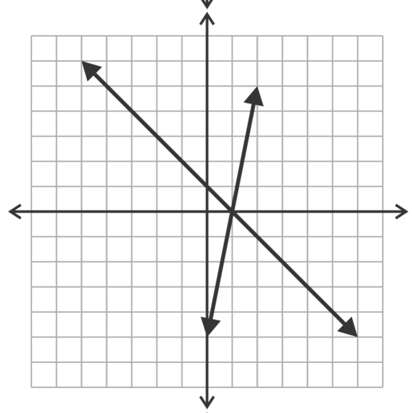

c.

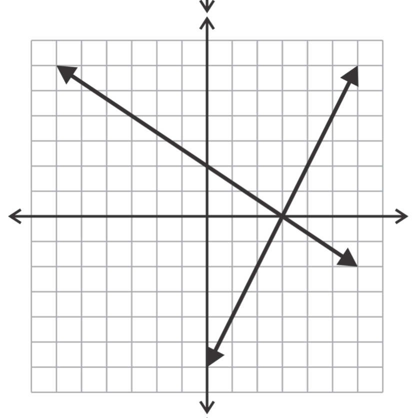

d.

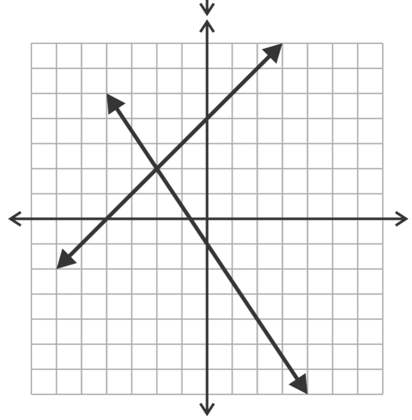
4.

$$y = 5x - 5$$
$$y = -x + 1$$

a. 

b. 

c. 

d. 

Solve the following linear systems by graphing. Use graph paper and a straightedge to insure accuracy. You are encouraged to verify your answer algebraically.

5.

$$y = -\frac{2}{5}x + 1$$
$$y = \frac{3}{5}x - 4$$

6.

$$y = -\frac{2}{3}x + 4$$
$$y = 3x - 7$$

7.

$$y = -2x + 1$$
$$x - y = -4$$

8.

$$3x + 4y = 12$$
$$x - 4y = 4$$

9.

$$7x - 2y = -4$$
$$y = -5$$

10.

$$x - 2y = -8$$
$$x = -3$$

Solve the following linear systems by graphing using technology. Solutions should be rounded to the nearest hundredth as necessary.

11.

$$y = \frac{3}{7}x + 11$$
$$y = -\frac{13}{2}x - 5$$

12.

$$y = 0.95x - 8.3$$
$$2x + 9y = 23$$

13.

$$15x - y = 22$$
$$3x + 8y = 15$$

Use the following information to complete exercises 14-17.

Clara and her brother, Carl, are at the beach for vacation. They want to rent bikes to ride up and down the boardwalk. One rental shop, Bargain Bikes, advertises rates of $5 plus $1.50 per hour. A second shop, Frugal Wheels, advertises a rate of $6 plus $1.25 an hour.

14. How much does it cost to rent a bike for one hour from each shop? How about 10 hours?
15. Write equations to represent the cost of renting a bike from each shop. Let x represent the number of hours and y represent the total cost.
16. Solve your system to figure out when the cost is the same.
17. Clara and Carl want to rent the bikes for about 3 hours. Which shop should they use?

5.2 Substitution Method for Systems of Equations

Here you'll learn how to solve systems of linear equations algebraically using the substitution method.

When you solve a system of consistent and independent equations by graphing, a single ordered pair is the solution. The ordered pair satisfies both equations and the point is the intersection of the graphs of the linear equations. The coordinates of this point of intersection are not always integers. How can you algebraically solve a system of equations like the one below?

$$\left\{\begin{array}{c} 2x+3y=13 \\ y=4x-5 \end{array}\right\}$$

Guidance

A 2×2 system of linear equations can be solved algebraically by the substitution method. In order to use this method, follow these steps:

1. Solve one of the equations for one of the variables.
2. Substitute that expression into the remaining equation. The result will be a linear equation, with one variable, that can be solved.
3. Solve the remaining equation.
4. Substitute the solution into the other equation to determine the value of the other variable.
5. The solution to the system is the intersection point of the two equations and it represents the coordinates of the ordered pair.

Example A

Solve the following system of linear equations by substitution:

$$\left\{\begin{array}{c} 3x+y=1 \\ 2x+5y=18 \end{array}\right\}$$

Solution: To begin, solve one of the equations in terms of one of the variables. This step is simplified if one of the equations has one variable with a coefficient that is either +1 or –1. In the above system the first equation has 'y' with a coefficient of 1.

$$\begin{aligned} &3x+y=1 \\ &3x-3x+y=1-3x \\ &\boxed{y=1-3x} \end{aligned}$$

Substitute $(1-3x)$ into the second equation for 'y'.

$$\begin{aligned}2x+5y&=18\\2x+5(1-3x)&=18\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}&2x+5-15x=18\\&-13x+5=18\\&-13x+5-5=18-5\\&-13x=13\\&\frac{-13x}{-13}=\frac{13}{-13}\\&\frac{\cancel{-13}x}{\cancel{-13}}=\frac{\overset{-1}{\cancel{-13}}}{\cancel{-13}}\\&\boxed{x=-1}\end{aligned}$$

Substitute –1 for x into the equation

$$\boxed{y=1-3x}$$

.

$$\begin{aligned}y&=1-3x\\y&=1-3(-1)\end{aligned}$$

$$\begin{aligned}y&=1+3\\&\boxed{y=4}\end{aligned}$$

The solution is (–1, 4). This represents the intersection point of the lines if the equations were graphed on a Cartesian grid. Another way to write 'the lines intersect at (–1, 4)' is:

$$\boxed{\text{Line1}}:3x+y=1$$

$$\boxed{\text{Line2}}:2x+5y=18$$

Line 1 intersects Line 2 at (–1, 4)

$$\begin{array}{c}\text{at}\\\uparrow\\\boxed{l_1\cap l_2@(-1,4)}\\\downarrow\\\text{intersects}\end{array}$$

Example B

Solve the following system of linear equations by substitution:

$$\left\{\begin{matrix} 8x-3y=6 \\ 6x+12y=-24 \end{matrix}\right\}$$

Solution: There is no variable that has a coefficient of +1 or of –1. However, the second equation has coefficients and a constant that are multiples of 6. The second equation will be solved for the variable 'x'.

$$\begin{aligned}
&6x+12y=-24\\
&6x+12y-12y=-24-12y\\
&6x=-24-12y\\
&\frac{6x}{6}=\frac{-24}{6}-\frac{12y}{6}\\
&\frac{\cancel{6}x}{\cancel{6}}=\frac{\overset{-4}{\cancel{-24}}}{\cancel{6}}-\frac{\overset{2}{\cancel{12}}y}{\cancel{6}}\\
&\boxed{x=-4-2y}
\end{aligned}$$

Substitute $(-4-2y)$ into the first equation for 'x'.

$$\begin{aligned}
8x-3y&=6\\
8(-4-2y)-3y&=6
\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}
&-32-16y-3y=6\\
&-32-19y=6\\
&-32+32-19y=6+32\\
&-19y=38\\
&\frac{-19y}{-19}=\frac{38}{-19}\\
&\frac{\cancel{-19}y}{\cancel{-19}}=\frac{\overset{-2}{\cancel{38}}}{\cancel{-19}}\\
&\boxed{y=-2}
\end{aligned}$$

Substitute –2 for y into the equation

$$\boxed{x=-4-2y}$$

$$\begin{aligned}
x&=-4-2y\\
x&=-4-2(-2)
\end{aligned}$$

$$x = -4 + 4$$
$$\boxed{x = 0}$$
$$\boxed{l_1 \cap l_2 @ (0, -2)}$$

Example C

Jason, who is a real computer whiz, decided to set up his own server and to sell space on his computer so students could have their own web pages on the Internet. He devised two plans. One plan charges a base fee of \$25.00 plus \$0.50 every month. His other plan has a base fee of \$5.00 plus \$1 per month.

i) Write an equation to represent each plan.

ii) Solve the system of equations.

Solution: Both plans deal with the cost of buying space from Jason's server. The cost involves a base fee and a monthly rate. The equations for the plans are:

- $y = 0.50x + 25$
- $y = 1x + 5$

where 'y' represents the **cost** and 'x' represents the **number of months.** Both equations are equal to 'y'. Therefore, the expression for y can be substituted for the y in the other equation.

$$\left\{\begin{matrix} y = 0.50x + 25 \\ y = 1x + 5 \end{matrix}\right\}$$

$$0.50x + 25 = 1x + 5$$
$$0.50x + 25 - 25 = 1x + 5 - 25$$
$$0.50x = 1x - 20$$
$$0.50x - 1x = 1x - 1x - 20$$
$$-0.50x = -20$$
$$\frac{-0.50x}{-0.50} = \frac{-20}{-0.50}$$
$$\frac{\cancel{-0.50}x}{\cancel{-0.50}} = \frac{\overset{40}{\cancel{-20}}}{\cancel{-0.50}}$$
$$\boxed{x = 40 \text{ months}}$$

Since the equations were equal, the value for 'x' can be substituted into either of the original equations. The result will be the same.

$$y = 1x + 5$$
$$y = 1(40) + 5$$

$$y = 40 + 5$$
$$\boxed{y = 45\ dollars}$$
$$\boxed{l_1 \cap l_2 @ (40, 45)}.$$

Concept Problem Revisited

When graphing is not a feasible method for solving a system, you can solve by substitution:

$$\begin{Bmatrix} 2x + 3y = 13 \\ y = 4x - 5 \end{Bmatrix}$$

The second equation is solved in terms of the variable 'y'. The expression $(4x + 5)$ can be used to replace 'y' in the first equation.

$$2x + 3y = 13$$
$$2x + 3(4x - 5) = 13$$

The equation now has one variable. Apply the distributive property.

$2x + 12x - 15 = 13$

Combine like terms to simplify the equation.

$14x - 15 = 13$

Solve the equation.

$$14x - 15 + 15 = 13 + 15$$
$$14x = 28$$
$$\frac{14x}{14} = \frac{28}{14}$$
$$\frac{\cancel{14}x}{\cancel{14}} = \frac{\overset{2}{\cancel{28}}}{\cancel{14}}$$
$$\boxed{x = 2}$$

To determine the value of 'y', substitute this value into the equation $y = 4x - 5$.

$$y = 4x - 5$$
$$y = 4(2) - 5$$

$$y = 8 - 5$$
$$\boxed{y = 3}$$

The solution is (2, 3). This represents the intersection point of the lines if the equations were graphed on a Cartesian grid.

Vocabulary

Substitution Method

The ***substitution method*** is a way of solving a system of linear equations algebraically. The substitution method involves solving an equation for a variable and substituting that expression into the other equation.

Guided Practice

1. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} x = 2y + 1 \\ x = 4y - 3 \end{Bmatrix}$$

2. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} 2x + y = 3 \\ 3x + 2y = 12 \end{Bmatrix}$$

.

3. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} \frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\ -\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4} \end{Bmatrix}$$

Answers:

1.

$$\begin{Bmatrix} x = 2y + 1 \\ x = 4y - 3 \end{Bmatrix}$$

Both equations are equal to the variable 'x'. Set $(2y+1) = (4y-3)$
$2y + 1 = 4y - 3$
Solve the equation.

$$\begin{aligned}
&2y + 1 - 1 = 4y - 3 - 1 \\
&2y = 4y - 4 \\
&2y - 4y = 4y - 4y - 4 \\
&-2y = -4 \\
&\frac{-2y}{-2} = \frac{-4}{-2} \\
&\frac{\cancel{-2}y}{\cancel{-2}} = \frac{\overset{2}{\cancel{-4}}}{\cancel{-2}} \\
&\boxed{y = 2}
\end{aligned}$$

Substitute this value for 'y' into one of the original equations.

$$\begin{aligned}
x &= 2y + 1 \\
x &= 2(2) + 2
\end{aligned}$$

$$x = 4 + 1$$
$$\boxed{x = 5}$$
$$\boxed{l_1 \cap l_2 @ (5,2)}$$

2.

$$\left\{ \begin{array}{c} 2x + y = 3 \\ 3x + 2y = 12 \end{array} \right\}$$

The first equation has the variable 'y' with a coefficient of 1. Solve the equation in terms of 'y'.

$$2x + y = 3$$
$$2x - 2x + y = 3 - 2x$$
$$\boxed{y = 3 - 2x}$$

Substitute $(3 - 2x)$ into the second equation for 'y'.

$$3x + 2y = 12$$
$$3x + 2(3 - 2x) = 12$$

Apply the distributive property and solve the equation.

$$3x + 6 - 4x = 12$$
$$6 - x = 12$$
$$6 - 6 - x = 12 - 6$$
$$-x = 6$$
$$\frac{-x}{-1} = \frac{6}{-1}$$
$$\frac{\cancel{-1}x}{\cancel{-1}} = \frac{\overset{-6}{\cancel{6}}}{\cancel{-1}}$$
$$\boxed{x = -6}$$

Substitute this value for 'x' into the equation $y = 3 - 2x$.

$$y = 3 - 2x$$
$$y = 3 - 2(-6)$$

$$y = 3 + 12$$
$$\boxed{y = 15}$$
$$\boxed{l_1 \cap l_2 @ (-6,15)}$$

3.

$$\begin{Bmatrix} \frac{2}{5}m+\frac{3}{4}n=\frac{5}{2} \\ -\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4} \end{Bmatrix}$$

Begin by multiplying each equation by the LCM of the denominators to simplify the system.
$\frac{2}{5}m+\frac{3}{4}n=\frac{5}{2}$ The LCM for the denominators is 20.

$$\begin{aligned} 20\left(\frac{2}{5}\right)m+20\left(\frac{3}{4}\right)n &= 20\left(\frac{5}{2}\right) \\ \overset{4}{\cancel{20}}\left(\frac{2}{\cancel{5}}\right)m+\overset{5}{\cancel{20}}\left(\frac{3}{\cancel{4}}\right)n &= \overset{10}{\cancel{20}}\left(\frac{5}{\cancel{2}}\right) \\ 8m+15n &= 50 \\ 8m+15n &= 50 \end{aligned}$$

$-\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4}$ The LCM for the denominators is 12.

$$\begin{aligned} -12\left(\frac{2}{3}\right)m+12\left(\frac{1}{2}\right)n &= 12\left(\frac{3}{4}\right) \\ -\overset{4}{\cancel{12}}\left(\frac{2}{\cancel{3}}\right)m+\overset{6}{\cancel{12}}\left(\frac{1}{\cancel{2}}\right)n &= \overset{3}{\cancel{12}}\left(\frac{3}{\cancel{4}}\right) \\ -8m+6n &= 9 \\ -8m+6n &= 9 \end{aligned}$$

The two equations that need to be solved by substitution are:

$$\begin{Bmatrix} 8m+15n=50 \\ -8m+6n=9 \end{Bmatrix}$$

Neither of the equations have a variable with a coefficient of 1 nor does one equation have coefficients that are multiples of a given coefficient. Solve the first equation in terms of 'm'.

$$\begin{aligned} &8m+15n=50 \\ &8m+15n-15n=50-15n \\ &8m=50-15n \\ &\frac{8m}{8}=\frac{50}{8}-\frac{15n}{8} \\ &\frac{\cancel{8}m}{\cancel{8}}=\frac{50}{8}-\frac{15n}{8} \\ &m=\frac{25}{4}-\frac{15}{8}n \\ &\boxed{m=\frac{25}{4}-\frac{15}{8}n} \end{aligned}$$

Substitute this value for 'm' into the second equation.

$$\begin{aligned} -8m+6n &= 9 \\ -8\left(\frac{25}{4}-\frac{15}{8}n\right)+6n &= 9 \end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}
&-\frac{200}{4}+\frac{120}{8}n+6n=9\\
&-\frac{\overset{50}{\cancel{200}}}{\cancel{4}}+\frac{\overset{15}{\cancel{120}}}{\cancel{8}}n+6n=9\\
&-50+15n+6n=9\\
&-50+21n=9\\
&-50+50+21n=9+50\\
&21n=59\\
&\frac{21n}{21}=\frac{59}{21}\\
&\frac{\cancel{21}n}{\cancel{21}}=\frac{59}{21}\\
&\boxed{n=\frac{59}{21}}
\end{aligned}$$

Substitute this value into the equation that has been solved in terms of 'm' or into one of the original equations or into one of the new equations that resulted from multiplying by the LCM.

Whichever substitution is performed, the same result will occur.

$$\begin{aligned}
m&=\frac{25}{4}-\frac{15}{8}n\\
m&=\frac{25}{4}-\frac{15}{8}\left(\frac{59}{21}\right)\\
m&=\frac{25}{4}-\frac{885}{168}
\end{aligned}$$

A common denominator is required to subtract the fractions.

$$\begin{array}{r}
42\\
4\overline{)168}\\
\underline{-16\downarrow}\\
8\\
\underline{-\ \ 8}\\
0
\end{array}$$

Multiply $\frac{25}{4}$ *by* $\frac{42}{42}$:

$$m = \frac{42}{42}\left(\frac{25}{4}\right) - \frac{885}{168}$$
$$m = \frac{1050}{168} - \frac{885}{168}$$
$$m = \frac{165}{168}$$
$$m = \frac{55}{56}$$
$$\boxed{m = \frac{55}{56}}$$
$$\boxed{l_1 \cap l_2 @ \left(\frac{55}{56}, \frac{59}{21}\right)}$$

Practice

Solve the following systems of linear equations using the substitution method.

1. $$\left\{\begin{matrix} y = 3x \\ 5x - 2y = 1 \end{matrix}\right\}$$

2. $$\left\{\begin{matrix} y = 3x + 1 \\ 2x - y = 2 \end{matrix}\right\}$$

3. $$\left\{\begin{matrix} x = 2y \\ x = 3y - 3 \end{matrix}\right\}$$

4. $$\left\{\begin{matrix} x - y = 6 \\ 6x - y = 40 \end{matrix}\right\}$$

5. $$\left\{\begin{matrix} x + y = 6 \\ x + 3(y + 2) = 10 \end{matrix}\right\}$$

6.

$$\begin{Bmatrix} 2x+y=5 \\ 3x-4y=2 \end{Bmatrix}$$

7.

$$\begin{Bmatrix} 5x-2y=-4 \\ 4x+y=-11 \end{Bmatrix}$$

8.

$$\begin{Bmatrix} 3y-x=-10 \\ 3x+4y=-22 \end{Bmatrix}$$

9.

$$\begin{Bmatrix} 4e+2f=-2 \\ 2e-3f=1 \end{Bmatrix}$$

10.

$$\begin{Bmatrix} \frac{1}{4}x+y=-\frac{7}{2} \\ \frac{1}{2}x-\frac{1}{4}y=1 \end{Bmatrix}$$

11.

$$\begin{Bmatrix} x=-4+y \\ x=3y-6 \end{Bmatrix}$$

12.

$$\begin{Bmatrix} 3y-2x=-3 \\ 3x-3y=6 \end{Bmatrix}$$

13.

$$\begin{Bmatrix} 2x=5y-12 \\ 3x+5y=7 \end{Bmatrix}$$

14.

$$\begin{Bmatrix} 3y = 2x - 5 \\ 2x = y + 3 \end{Bmatrix}$$

15.

$$\begin{Bmatrix} \frac{x+y}{3} + \frac{x-y}{2} = \frac{25}{6} \\ \frac{x+y-9}{2} = \frac{y-x-6}{3} \end{Bmatrix}$$

5.3 Elimination Method for Systems of Equations

Here you'll learn how to solve a system of linear equations algebraically using the elimination method.

When you solve a system of consistent and independent equations by graphing, a single ordered pair is the solution. The ordered pair satisfies both equations and the point is the intersection of the graphs of the linear equations. The coordinates of this point of intersection are not always integers. Therefore, some method has to be used to determine the values of the coordinates. How can you algebraically solve the system of equations below without first rewriting the equations?

$$\left\{\begin{array}{c} 2x+3y=5 \\ 3x-3y=10 \end{array}\right\}$$

Guidance

A 2×2 system of linear equations can be solved algebraically by the elimination method. To use this method you must write an equivalent system of equations such that, when two of the equations are added or subtracted, one of the variables is eliminated. The solution is the intersection point of the two equations and it represents the coordinates of the ordered pair. This method is demonstrated in the examples.

Example A

Solve the following system of linear equations by elimination:

$$\left\{\begin{array}{c} 3y=2x-5 \\ 2x=y+3 \end{array}\right\}$$

Solution: To begin, set up the equations so that they are in the format

$$\left\{\begin{array}{c} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{array}\right\}$$

$$\begin{aligned} 3y&=2x-5 \\ -2x+3y&=2x-2x-5 \\ -2x+3y&=-5 \end{aligned} \qquad\qquad \begin{aligned} 2x&=y+3 \\ 2x-y&=y-y+3 \\ 2x-y&=3 \end{aligned}$$

Solve the formatted system of equations:

$$\left\{\begin{array}{c} -2x+3y=-5 \\ 2x-y=3 \end{array}\right\}$$

Both equations have a term that is $2x$. In the first equation the coefficient of x is a negative two and in the second equation the coefficient of x is a positive two. If the two equations are added, the x variable is eliminated.

$$\begin{aligned}
-\cancel{2x}+3y&=-5\\
\underline{\cancel{2x}-\ y}&\underline{=+3}\\
2y&=-2 \quad \text{Eliminate the variable } x.\\
2y&=-2 \quad \text{Solve the equation.}\\
\frac{2y}{2}&=\frac{-2}{2}\\
\frac{\cancel{2}y}{\cancel{2}}&=\frac{\overset{-1}{\cancel{-2}}}{\cancel{2}}\\
&\boxed{y=-1}
\end{aligned}$$

The value of y is –1. This value can now be substituted into one of the original equations to determine the value of x. Remember x is the variable that was eliminated from the system of linear equations.

$$\begin{aligned}
&2x-y=3\\
&2x-(-1)=5 && \text{Substitute in the value for } y.\\
&2x+1=5 && \text{Multiply the value of } x \text{ by the coefficient } (-1).\\
&2x+1-1=3-1 && \text{Isolate the variable } x.\\
&2x=2 && \text{Solve the equation.}\\
&\frac{2x}{2}=\frac{2}{2}\\
&\frac{\cancel{2}x}{\cancel{2}}=\frac{\overset{1}{\cancel{2}}}{\cancel{2}}\\
&\boxed{x=1}\\
&\boxed{l_1\cap l_2 @ (1,-1)}
\end{aligned}$$

This means "Line 1 intersects Line 2 at the point (1, –1)".

Example B

Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} 2x-3y=13\\ 3x+4y=-6 \end{Bmatrix}$$

Solution: The coefficients of 'x' are 2 and 3. The coefficients of 'y' are –3 and 4. To eliminate a variable the coefficients must be the same number but with opposite signs. This can be accomplished by multiplying one or both of the equations.

The first step is to choose a variable to eliminate. If the choice is 'x', the least common multiple of 2 and 3 is 6. This means that the equations must be multiplied by 3 and 2 respectively. One of the multipliers must be a negative number so that one of the coefficients of 'x' will be a negative 6. When this is done, the coefficients of 'x' will be +6 and –6. The variable will then be eliminated when the equations are added.

Multiply the first equation by negative three.

$$-3(2x-3y=13)$$
$$-6x+9y=-39$$

Multiply the second equation by positive two.

$$2(3x+4y=-6)$$
$$6x+8y=-12$$

Add the two equations.

$$\begin{aligned} -\cancel{6x}+9y&=-39\\ \underline{\cancel{6x}+8y}&\underline{=-12}\\ 17y&=-51 \qquad \text{Solve the equation.}\\ 17y&=-51\\ \frac{17y}{17}&=\frac{-51}{17}\\ \frac{\cancel{17}y}{\cancel{17}}&=\frac{\overset{-3}{\cancel{-51}}}{\cancel{17}}\\ &\boxed{y=-3} \end{aligned}$$

Substitute the value for y into one of the original equations.

$$\begin{aligned} &2x-3y=13 && \\ &2x-3(-3)=13 && \text{Substitute in the value for } y.\\ &2x+9=13 && \text{Multiply the value of } y \text{ by the coefficient } (-3).\\ &2x+9-9=13-9 && \text{Isolate the variable } x.\\ &2x=4 && \text{Solve the equation.}\\ &\frac{2x}{2}=\frac{4}{2} && \\ &\frac{\cancel{2}x}{\cancel{2}}=\frac{\overset{2}{\cancel{4}}}{\cancel{2}} && \\ &\boxed{x=2} && \\ &\boxed{l_1\cap l_2 @ (2,-3)} && \end{aligned}$$

Example C

Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} \frac{3}{4}x+\frac{5}{4}y=4 \\ \frac{1}{2}x+\frac{1}{3}y=\frac{5}{3} \end{Bmatrix}$$

Solution: Begin by multiplying each equation by the LCD to create two equations with integers as the coefficients of the variables.

$$\begin{aligned} \frac{3}{4}x+\frac{5}{4}y&=4 \\ 4\left(\frac{3}{4}\right)x+4\left(\frac{5}{4}\right)y&=4(4) \\ \cancel{4}\left(\frac{3}{\cancel{4}}\right)x+\cancel{4}\left(\frac{5}{\cancel{4}}\right)y&=4(4) \\ 3x+5y&=16 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}x+\frac{1}{3}y&=\frac{5}{3} \\ 6\left(\frac{1}{2}\right)x+6\left(\frac{1}{3}\right)y&=6\left(\frac{5}{3}\right) \\ \overset{3}{\cancel{6}}\left(\frac{1}{\cancel{2}}\right)x+\overset{2}{\cancel{6}}\left(\frac{1}{\cancel{3}}\right)y&=\overset{2}{\cancel{6}}\left(\frac{5}{\cancel{3}}\right) \\ 3x+2y&=10 \end{aligned}$$

Now solve the following system of equations by elimination:

$$\begin{Bmatrix} 3x+5y=16 \\ 3x+2y=10 \end{Bmatrix}$$

The coefficients of the 'x' variable are the same –positive three. To change one of them to a negative three, multiply one of the equations by a negative one.

$$\begin{gathered} -1(3x+5y=16) \\ -3x-5y=-16 \end{gathered}$$

The two equations can now be added.

$$\begin{aligned} -\cancel{3x}-5y&=-16 \\ \underline{\cancel{3x}+2y} &\underline{=\quad 10} \\ -3y&=\ \ -6 \qquad \text{Solve the equation.} \\ -3y&=-6 \\ \frac{-3y}{-3}&=\frac{-6}{-3} \\ \frac{\cancel{-3}y}{\cancel{-3}}&=\frac{\overset{2}{\cancel{-6}}}{\cancel{-3}} \\ \boxed{y=2} \end{aligned}$$

Substitute the value for 'y' into one of the original equations.

$$\frac{3}{4}x+\frac{5}{4}y=4$$

$$\frac{3}{4}x+\frac{5}{4}(2)=4$$ Substitute in the value for y.

$$\frac{3}{4}x+\frac{10}{4}=4$$ Multiply the value of y by the coefficient $\left(\frac{5}{4}\right)$.

$$\frac{3}{4}x+\frac{10}{4}-\frac{10}{4}=4-\frac{10}{4}$$ Isolate the variable x.

$$\frac{3}{4}x=\frac{16}{4}-\frac{10}{4}$$

$$\frac{3}{4}x=\frac{6}{4}$$ Multiply both sides by 4.

$$4\left(\frac{3}{4}x\right)=4\left(\frac{6}{4}\right)$$

$$\cancel{4}\left(\frac{3}{\cancel{4}}x\right)=\cancel{4}\left(\frac{6}{\cancel{4}}\right)$$

$$3x=6$$ Solve the equation.

$$\frac{3x}{3}=\frac{6}{3}$$

$$\frac{\cancel{3}x}{\cancel{3}}=\frac{\overset{2}{\cancel{6}}}{\cancel{3}}$$

$$\boxed{x=2}$$

$$\boxed{l_1\cap l_2 @ (2,2)}$$

Concept Problem Revisited

Solve by elimination:

$$\begin{Bmatrix} 2x+3y=5 \\ 3x-3y=10 \end{Bmatrix}$$

Both equations have a term that is $3y$. In the first equation the coefficient of 'y' is a positive three and in the second equation the coefficient of 'y' is a negative three. If the two equations are added, the 'y' variable is eliminated.

$$\begin{aligned} 2x+3y&=5 \\ \underline{3x-3y}&\underline{=10} \\ 5x&=15 \end{aligned}$$

The resulting equation now has one variable. Solve this equation:

$$\begin{aligned}
5x &= 15 \\
\frac{5x}{5} &= \frac{15}{5} \\
\frac{\cancel{5}x}{\cancel{5}} &= \frac{\cancelto{3}{15}}{\cancel{5}} \\
&\boxed{x = 3}
\end{aligned}$$

The value of 'x' is 3. This value can now be substituted into one of the original equations to determine the value of 'y'.

$$\begin{aligned}
&2x+3y=5 && \\
&2(3)+3y=5 && \text{Substitute in the value for } x. \\
&6+3y=5 && \text{Multiply the value of } x \text{ by the coefficient } (2). \\
&6-6+3y=5-6 && \text{Isolate the variable } y. \\
&3y=-1 && \text{Solve the equation.} \\
&\frac{3y}{3}=\frac{-1}{3} && \\
&\frac{\cancel{3}y}{\cancel{3}}=-\frac{1}{3} && \\
&\boxed{y=-\frac{1}{3}} &&
\end{aligned}$$

The solution to the system of linear equations is $x = 3$ and $y = -\frac{1}{3}$. This solution means

$$\boxed{l_1 \cap l_2 @ \left(3, -\frac{1}{3}\right)}$$

Vocabulary

Elimination Method

The ***elimination method*** is a method used for solving a system of linear equations algebraically. This method involves obtaining an equivalent system of equations such that, when two of the equations are added or subtracted, one of the variables is eliminated.

Guided Practice

1. Solve the following system of linear equations by elimination:

$$\left\{\begin{array}{c} 4x-15y=5 \\ 6x-5y=4 \end{array}\right\}$$

2. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} 3x = 7y + 41 \\ 5x = 3y + 51 \end{Bmatrix}.$$

3. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} \frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\ -\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4} \end{Bmatrix}$$

Answers:

1.

$$\begin{Bmatrix} 4x - 15y = 5 \\ 6x - 5y = 4 \end{Bmatrix}$$

Multiply the second equation by (–3) to eliminate the variable 'y'.

$$\begin{aligned} &-3(6x - 5y = 4) \\ &-18x + 15y = -12 \\ &-18x + 15y = -12 \end{aligned}$$

Add the equations:

$$\begin{array}{r} 4x - \cancel{15y} = 5 \\ -18x + \cancel{15y} = -12 \\ \hline -14x = -7 \end{array}$$

Solve the equation:

$$\begin{aligned} &-14x = -7 \\ &\frac{-14x}{-14} = \frac{-7}{-14} \\ &\frac{\cancel{-14}x}{\cancel{-14}} = \frac{-7}{-14} \\ &\boxed{x = \frac{1}{2}} \end{aligned}$$

Substitute this value for 'x' into one of the original equations.

$$
\begin{aligned}
&4x - 15y = 5 \\
&4\left(\frac{1}{2}\right) - 15y = 5 && \text{Substitute in the value for } x. \\
&2 - 15y = 5 && \text{Multiply the value of } x \text{ by the coefficient } (4). \\
&2 - 2 - 15y = 5 - 2 && \text{Isolate the variable } y. \\
&-15y = 3 && \text{Solve the equation.} \\
&\frac{-15y}{-15} = \frac{3}{-15} \\
&\frac{\cancel{-15}y}{\cancel{-15}} = \frac{3}{-15} \\
&\boxed{y = -\frac{1}{5}} \\
&\boxed{l_1 \cap l_2 @ \left(\frac{1}{2}, -\frac{1}{5}\right)}
\end{aligned}
$$

2.

$$\begin{Bmatrix} 3x = 7y + 41 \\ 5x = 3y + 51 \end{Bmatrix}$$

Arrange the equations so that they are of the form

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{Bmatrix}$$

.

$$
\begin{aligned}
3x &= 7y + 41 & 5x &= 3y + 51 \\
3x - 7y &= 7y - 7y + 41 & 5x - 3y &= 3y - 3y + 51 \\
3x - 7y &= 41 & 5x - 3y &= 51
\end{aligned}
$$

Multiply the first equation by (–5) and the second equation by (3).

$$
\begin{aligned}
-5(3x - 7y &= 41) & 3(5x - 3y &= 51) \\
-15x + 35y &= -205 & 15x - 9y &= 153 \\
-15x + 35y &= -205 & 15x - 9y &= 153
\end{aligned}
$$

Add the equations to eliminate 'x'.

$$
\begin{aligned}
-\cancel{15x} + 35y &= -205 \\
\cancel{15x} - 9y &= 153 \\
\hline
26y &= -52
\end{aligned}
$$

Solve the equation:

$26y = 52$

$\frac{26y}{26} = \frac{-52}{26}$

$\frac{\cancel{26}y}{\cancel{26}} = \frac{\overset{-2}{\cancel{-52}}}{\cancel{26}}$

$\boxed{y = -2}$

$5x - 3y = 51$

$5x - 3(-2) = 51$ Substitute in the value for y.

$5x + 6 = 51$ Multiply the value of y by the coefficient (-3).

$5x + 6 - 6 = 51 - 6$ Isolate the variable x.

$5x = 45$ Solve the equation.

$\frac{5x}{5} = \frac{45}{5}$

$\frac{\cancel{5}x}{\cancel{5}} = \frac{\overset{9}{\cancel{45}}}{\cancel{5}}$

$\boxed{x = 9}$

$\boxed{l_1 \cap l_2 @ (9, -2)}$

3.

$$\left\{\begin{array}{c} \frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\ -\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4} \end{array}\right\}$$

Begin by multiplying each equation by the LCM of the denominators to simplify the system.
$\frac{2}{5}m + \frac{3}{4}n = \frac{5}{2}$ The LCM for the denominators is 20.

$20\left(\frac{2}{5}\right)m + 20\left(\frac{3}{4}\right)n = 20\left(\frac{5}{2}\right)$

$\overset{4}{\cancel{20}}\left(\frac{2}{\cancel{5}}\right)m + \overset{5}{\cancel{20}}\left(\frac{3}{\cancel{4}}\right)n = \overset{10}{\cancel{20}}\left(\frac{5}{\cancel{2}}\right)$

$8m + 15n = 50$

$\boxed{8m + 15n = 50}$

$-\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4}$ The LCM for the denominators is 12.

$-12\left(\frac{2}{3}\right)m + 12\left(\frac{1}{2}\right)n = 12\left(\frac{3}{4}\right)$

$-\overset{4}{\cancel{12}}\left(\frac{2}{\cancel{3}}\right)m + \overset{6}{\cancel{12}}\left(\frac{1}{\cancel{2}}\right)n = \overset{3}{\cancel{12}}\left(\frac{3}{\cancel{4}}\right)$

$-8m + 6n = 9$

$\boxed{-8m + 6n = 9}$

The two equations that need to be solved are:

$$\left\{\begin{array}{c} 8m + 15n = 50 \\ -8m + 6n = 9 \end{array}\right\}$$

The equations will be solved by using the elimination method. The variable 'm' has the same numerical coefficient with opposite signs. The variable will be eliminated when the equations are added.

$$\begin{array}{r} \cancel{8m}+15n=50 \\ \underline{-\cancel{8m}+6n=9} \\ 21n=59 \end{array}$$

Solve the equation:

$$21n=59$$
$$\frac{21n}{21}=\frac{59}{21}$$
$$\frac{\cancel{21}n}{\cancel{21}}=\frac{59}{21}$$
$$\boxed{n=\frac{59}{21}}$$
$$\frac{2}{5}m+\frac{3}{4}n=\frac{5}{2}$$
$$\frac{2}{5}m+\frac{3}{4}\left(\frac{59}{21}\right)=\frac{5}{2}$$

Substitute in the value for n.

$$\frac{2}{5}m+\frac{177}{84}=\frac{5}{2}$$

Multiply the value of y by the coefficient $\left(\frac{59}{21}\right)$.

$$\frac{2}{5}m+\frac{177}{84}-\frac{177}{84}=\frac{5}{2}-\frac{177}{84}$$

Isolate the variable x.

$$\frac{2}{5}m=\frac{210}{84}-\frac{177}{84}$$
$$\frac{2}{5}m=\frac{33}{84}$$

Solve the equation.

$$420\left(\frac{2}{5}\right)x=420\left(\frac{33}{84}\right)$$
$$\overset{84}{\cancel{420}}\left(\frac{2}{\cancel{5}}\right)x=\overset{5}{\cancel{420}}\left(\frac{33}{\cancel{84}}\right)$$
$$168x=165$$
$$\frac{168x}{168}=\frac{165}{168}$$
$$\frac{\cancel{168}x}{\cancel{168}}=\frac{165}{168}$$
$$\boxed{x=\frac{55}{56}}$$
$$\boxed{l_1\cap l_2 @\left(\frac{55}{56},\frac{59}{21}\right)}$$

Practice

Solve the following systems of linear equations using the elimination method.

1. $$\left\{\begin{matrix} 16x - y - 181 = 0 \\ 19x - y = 214 \end{matrix}\right\}$$

2. $$\left\{\begin{matrix} 3x + 2y + 9 = 0 \\ 4x = 3y + 5 \end{matrix}\right\}$$

3. $$\left\{\begin{matrix} x = 7y + 38 \\ 14y = -x - 46 \end{matrix}\right\}$$

4. $$\left\{\begin{matrix} 2x + 9y = -1 \\ 4x + y = 15 \end{matrix}\right\}$$

5. $$\left\{\begin{matrix} x - \frac{3}{5}y = \frac{26}{5} \\ 4y = 61 - 7x \end{matrix}\right\}$$

6. $$\left\{\begin{matrix} 3x - 5y = 12 \\ 2x + 10y = 4 \end{matrix}\right\}$$

7. $$\left\{\begin{matrix} 3x + 2y + 9 = 0 \\ 4x = 3y + 5 \end{matrix}\right\}$$

8. $$\left\{\begin{matrix} x = 69 + 6y \\ 3x = 4y - 45 \end{matrix}\right\}$$

9.

$$\begin{Bmatrix} 3(x-1)-4(y+2)=-5 \\ 4(x+5)-(y-1)=16 \end{Bmatrix}$$

10.

$$\begin{Bmatrix} \frac{3}{4}x-\frac{2}{5}y=2 \\ \frac{1}{7}x+\frac{3}{2}y=\frac{113}{7} \end{Bmatrix}$$

11.

$$\begin{Bmatrix} 3x+5y=17 \\ 2x+3y=11 \end{Bmatrix}$$

12.

$$\begin{Bmatrix} 3x-5y=-29 \\ 2x-8y=-42 \end{Bmatrix}$$

13.

$$\begin{Bmatrix} 7x-8y=-26 \\ 5x-12y=-45 \end{Bmatrix}$$

14.

$$\begin{Bmatrix} 6x+5y=5.1 \\ 4x-2y=-1.8 \end{Bmatrix}$$

15. When does it make sense to use the elimination method to solve a system of equations?

5.4 Consistent and Inconsistent Linear Systems

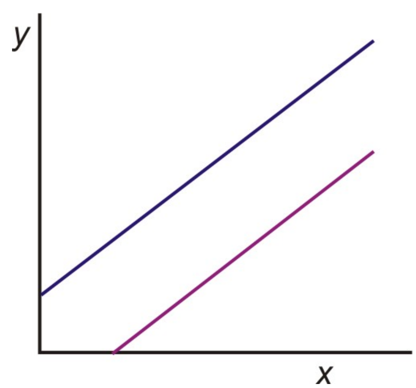
Here you'll learn the difference between three special types of linear systems: **inconsistent linear systems**, **consistent linear systems**, and **dependent linear systems**. You'll then use that information to determine the number of solutions a system has.

What if you were given a system of equations like $2x - y = 5$ and $10x - 5y = 25$? How could you rewrite these equations to determine the number of solutions the system has? After completing this Concept, you'll be able to identify whether a system of equations like this one is an inconsistent one, a consistent one, or a dependent one.

Guidance

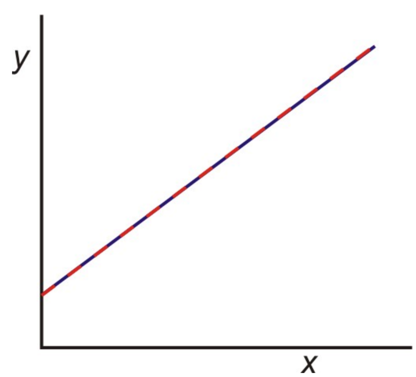
As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Or at least that's what usually happens. But what if the lines turn out to be parallel when we graph them?



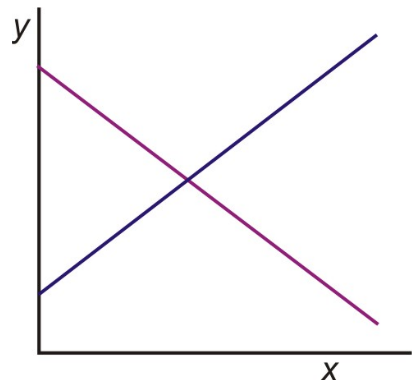
If the lines are parallel, they won't ever intersect. That means that the system of equations they represent has no solution. A system with no solutions is called an **inconsistent system.**

And what if the lines turn out to be identical?



If the two lines are the same, then *every* point on one line is also on the other line, so every point on the line is a solution to the system. The system has an **infinite number** of solutions, and the two equations are really just different forms of the same equation. Such a system is called a **dependent system**.

But usually, two lines cross at exactly one point and the system has exactly one solution:



A system with exactly one solution is called a **consistent system**.

To identify a system as **consistent, inconsistent**, or **dependent**, we can graph the two lines on the same graph and see if they intersect, are parallel, or are the same line. But sometimes it is hard to tell whether two lines are parallel just by looking at a roughly sketched graph.

Another option is to write each line in slope-intercept form and compare the slopes and $y-$ intercepts of the two lines. To do this we must remember that:

- Lines with different slopes always intersect.
- Lines with the same slope but different $y-$intercepts are parallel.
- Lines with the same slope and the same $y-$intercepts are identical.

Example A

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} 2x-5y &= 2 \\ 4x+y &= 5 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$2x-5y=2 \Rightarrow -5y=-2x+2 \Rightarrow y=\frac{2}{5}x-\frac{2}{5}$

$4x+y=5 \Rightarrow y=-4x+5$

The slopes of the two equations are different; therefore the lines must cross at a single point and the system has exactly one solution. This is a **consistent system.**

Example B

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} 3x &= 5-4y \\ 6x+8y &= 7 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$3x = 5 - 4y \Rightarrow 4y = -3x + 5 \Rightarrow y = -\frac{3}{4}x + \frac{5}{4}$

$6x + 8y = 7 \Rightarrow 8y = -6x + 7 \Rightarrow y = -\frac{3}{4}x + \frac{7}{8}$

The slopes of the two equations are the same but the $y-$intercepts are different; therefore the lines are parallel and the system has no solutions. This is an **inconsistent system.**

Example C

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x + y &= 3 \\ 3x + 3y &= 9 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$x + y = 3 \Rightarrow y = -x + 3$

$3x + 3y = 9 \Rightarrow 3y = -3x + 9 \Rightarrow y = -x + 3$

The lines are identical; therefore the system has an infinite number of solutions. It is a **dependent system.**

Vocabulary

- A system with no solutions is called an **inconsistent system.** For linear equations, this occurs with parallel lines.
- A system where the two equations overlap at one, multiple, or infinitely many points is called a **consistent system.**
- **Coincident lines** are lines with the same slope and $y-$intercept. The lines completely overlap.
- When solving a system of coincident lines, the resulting equation will be without variables and the statement will be true. You can conclude the system has an infinite number of solutions. This is called a **consistent-dependent system**.

Guided Practice

Determine whether the following system of linear equations has zero, one, or infinitely many solutions:

$$\begin{cases} 2y + 6x = 20 \\ y = -3x + 7 \end{cases}$$

What kind of system is this?

Solution:

It is easier to compare equations when they are in the same form. We will rewrite the first equation in slope-intercept form.

$2y+6x=20 \Rightarrow y+3x=10 \Rightarrow y=-3x+10$

Since the two equations have the same slope, but different y-intercepts, they are different but parallel lines. Parallel lines never intersect, so they have no solutions.

Since the lines are parallel, it is an inconsistent system.

Practice

Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.

1. $$\begin{aligned}3x-4y&=13\\ y&=-3x-7\end{aligned}$$
2. $$\begin{aligned}\frac{3}{5}x+y&=3\\ 1.2x+2y&=6\end{aligned}$$
3. $$\begin{aligned}3x-4y&=13\\ y&=-3x-7\end{aligned}$$
4. $$\begin{aligned}3x-3y&=3\\ x-y&=1\end{aligned}$$
5. $$\begin{aligned}0.5x-y&=30\\ 0.5x-y&=-30\end{aligned}$$
6. $$\begin{aligned}4x-2y&=-2\\ 3x+2y&=-12\end{aligned}$$
7. $$\begin{aligned}3x+y&=4\\ y&=5-3x\end{aligned}$$
8. $$\begin{aligned}x-2y&=7\\ 4y-2x&=14\end{aligned}$$

9.

$$\begin{aligned}-2y+4x&=8\\y-2x&=-4\end{aligned}$$

10.

$$\begin{aligned}x-\frac{y}{2}&=\frac{3}{2}\\3x+y&=6\end{aligned}$$

11.

$$\begin{aligned}0.05x+0.25y&=6\\x+y&=24\end{aligned}$$

12.

$$\begin{aligned}x+\frac{2y}{3}&=6\\3x+2y&=2\end{aligned}$$

5.5 Systems of Linear Inequalities

Here you'll learn how to graph and solve a system of two or more linear inequalities. You'll also determine if such systems are consistent or inconsistent.

What if you were given a system of linear inequalities like $6x - 2y \geq 3$ and $2y - 3x \leq 7$? How could you determine its solution? After completing this Concept, you'll be able to find the solution region of systems of linear inequalities like this one.

Guidance

In the last chapter you learned how to graph a linear inequality in two variables. To do that, you graphed the equation of the straight line on the coordinate plane. The line was solid for $\leq$ or $\geq$ signs (where the equals sign is included), and the line was dashed for $<$or $>$signs (where the equals sign is not included). Then you shaded above the line (if the inequality began with $y >$ or $y \geq$) or below the line (if it began with $y <$ or $y \leq$).

In this section, we'll see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes.

Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example A

Solve the following system:

$$\begin{aligned} 2x + 3y &\leq 18 \\ x - 4y &\leq 12 \end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

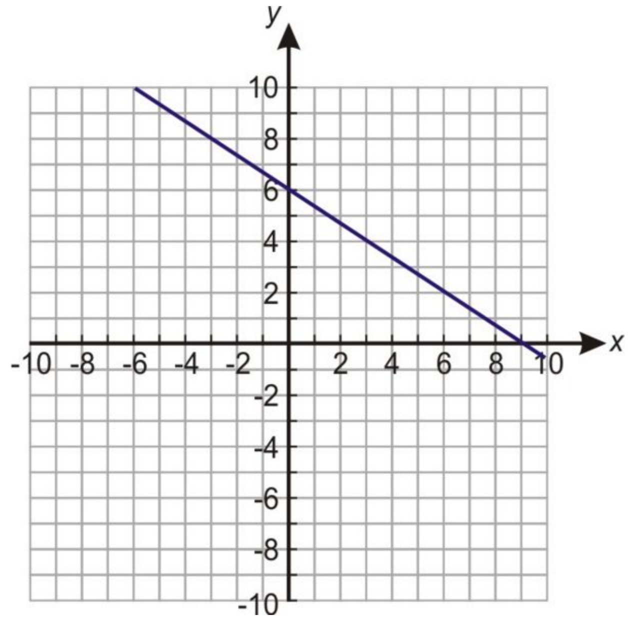
First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

$$\begin{aligned} 3y \leq -2x + 18 \qquad & \qquad y \leq -\frac{2}{3}x + 6 \\ & \Rightarrow \\ -4y \leq -x + 12 \qquad & \qquad y \geq \frac{x}{4} - 3 \end{aligned}$$

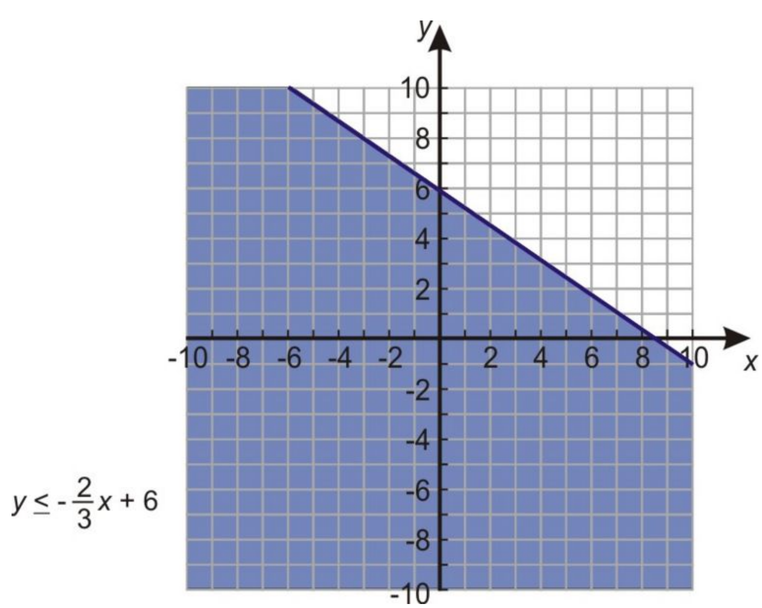
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

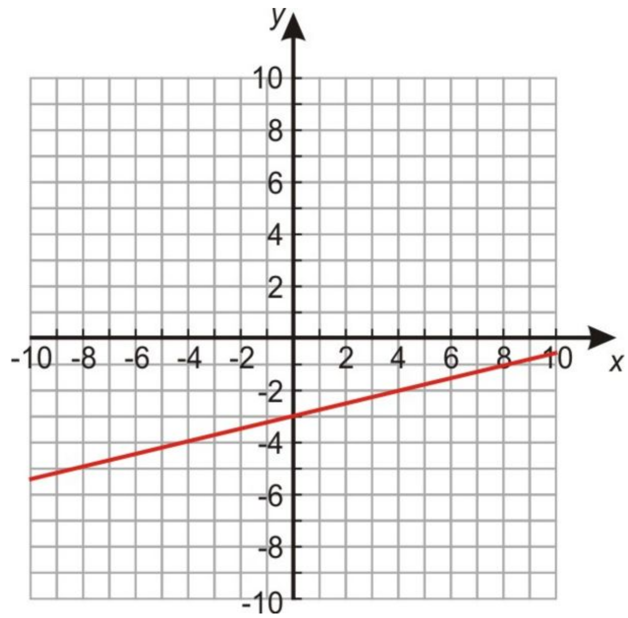
Here's the graph of the first inequality:



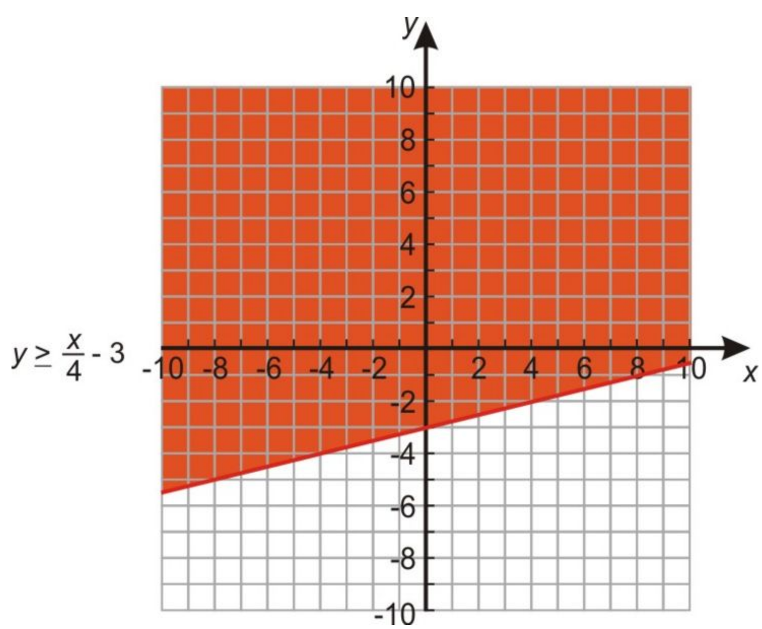
The line is solid because the equals sign is included in the inequality. Since the inequality is **less** than or equal to, we shade **below** the line.



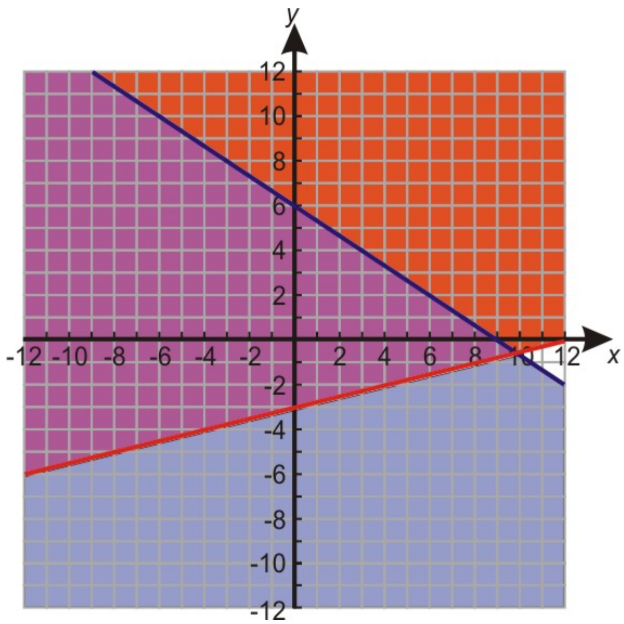
And here's the graph of the second inequality:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because y is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

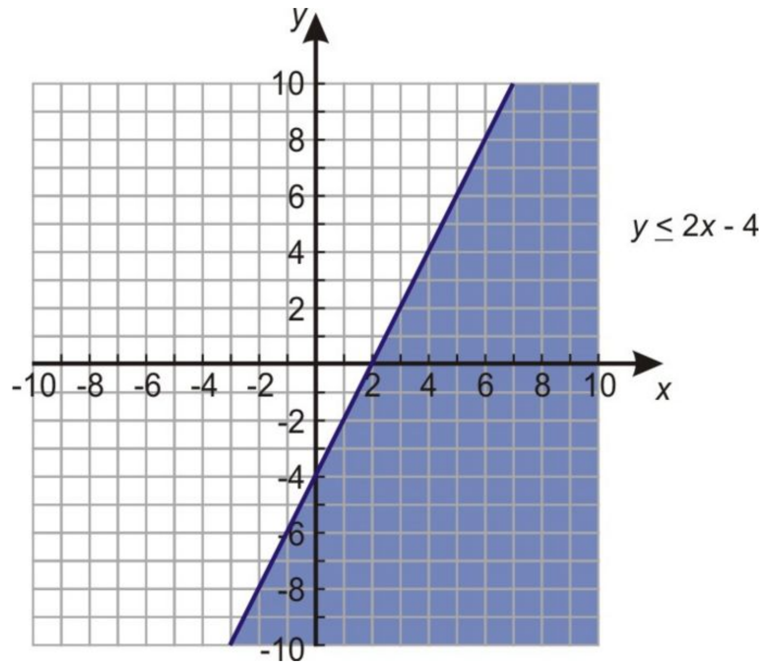
Example B

There are also situations where a system of inequalities has no solution. For example, let's solve this system.

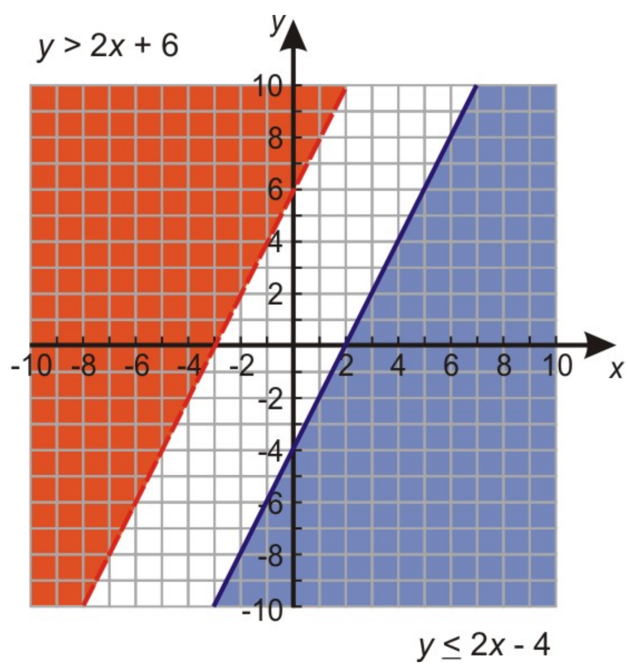
$$y \le 2x - 4$$
$$y > 2x + 6$$

Solution

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because y is greater than.



It doesn't look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won't ever overlap since the lines won't ever cross. So this system of inequalities has no solution.

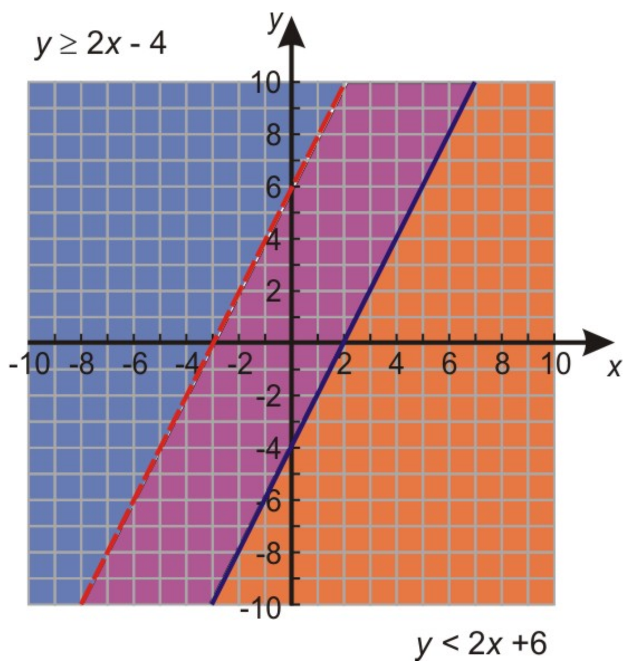
But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

To graph the system

$$y \geq 2x - 4$$
$$y < 2x + 6$$

,

we draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region—one that continues infinitely in at least one direction. But if we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded**—a finite region with three or more sides.

Let's look at a simple example.

Example C

Find the solution to the following system of inequalities.

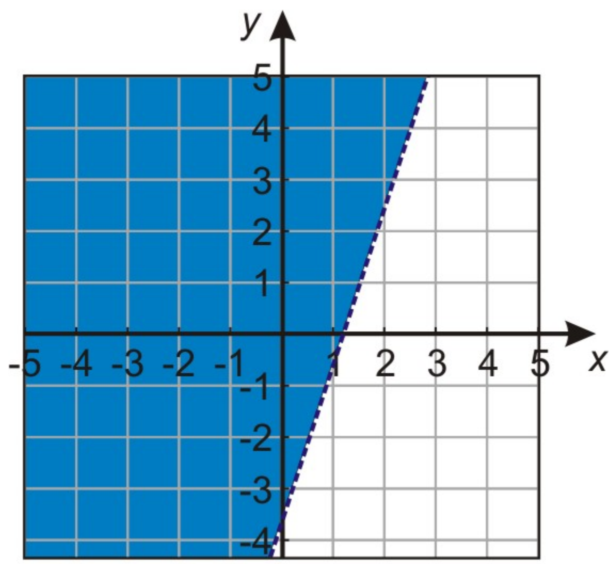
$$3x - y < 4$$
$$4y + 9x < 8$$
$$x \geq 0$$
$$y \geq 0$$

Solution

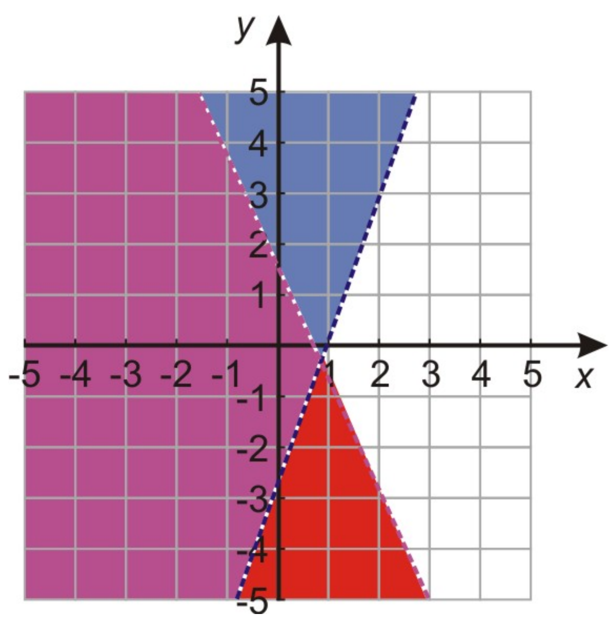
Let's start by writing our inequalities in slope-intercept form.

$$y > 3x - 4$$
$$y < -\frac{9}{4}x + 2$$
$$x \geq 0$$
$$y \geq 0$$

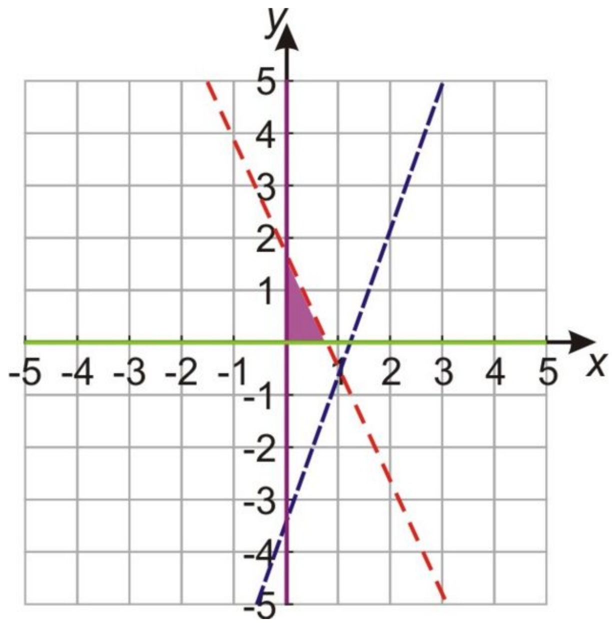
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$:



Next we graph $y < -\frac{9}{4}x + 2$:



Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.



The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x - 4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

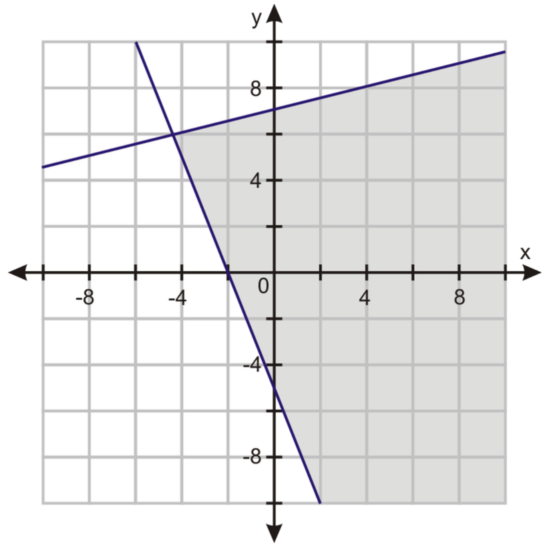
But that wasn't obvious until we actually drew the graph!

Vocabulary

- **Solution for the system of inequalities:** The ***solution for the system of inequalities*** is the common shaded region between all the inequalities in the system.
- **Feasible region:** The common shaded region of the system of inequalities is called the ***feasible region***.
- **Optimization:** The goal is to locate the feasible region of the system and use it to answer a profitability, or ***optimization***, question.

Guided Practice

Write the system of inequalities shown below.



Solution:

There are two boundary lines, so there are two inequalities. Write each one in slope-intercept form.

$$y \leq \frac{1}{4}x + 7$$
$$y \geq -\frac{5}{2}x - 5$$

Practice

1. Consider the system

$$y < 3x - 5$$
$$y > 3x - 5$$

. Is it consistent or inconsistent? Why?
2. Consider the system

$$y \leq 2x + 3$$
$$y \geq 2x + 3$$

. Is it consistent or inconsistent? Why?
3. Consider the system

$$y \leq -x + 1$$
$$y > -x + 1$$

. Is it consistent or inconsistent? Why?
4. In example 3 in this lesson, we solved a system of four inequalities and saw that one of the inequalities, $y > 3x - 4$, didn't affect the solution set of the system.
   a. What would happen if we changed that inequality to $y < 3x - 4$?
   b. What's another inequality that we could add to the original system without changing it? Show how by sketching a graph of that inequality along with the rest of the system.

c. What's another inequality that we could add to the original system to make it inconsistent? Show how by sketching a graph of that inequality along with the rest of the system.

5. Recall the compound inequalities in one variable that we worked with back in chapter 6. Compound inequalities with "and" are simply systems like the ones we are working with here, except with one variable instead of two.
   a. Graph the inequality $x > 3$ in two dimensions. What's another inequality that could be combined with it to make an inconsistent system?
   b. Graph the inequality $x \leq 4$ on a number line. What two-dimensional system would have a graph that looks just like this one?

Find the solution region of the following systems of inequalities.

6.

$$\begin{aligned} x-y &< -6 \\ 2y &\geq 3x+17 \end{aligned}$$

7.

$$\begin{aligned} 4y-5x &< 8 \\ -5x &\geq 16-8y \end{aligned}$$

8.

$$\begin{aligned} 5x-y &\geq 5 \\ 2y-x &\geq -10 \end{aligned}$$

9.

$$\begin{aligned} 5x+2y &\geq -25 \\ 3x-2y &\leq 17 \\ x-6y &\geq 27 \end{aligned}$$

10.

$$\begin{aligned} 2x-3y &\leq 21 \\ x+4y &\leq 6 \\ 3x+y &\geq -4 \end{aligned}$$

11.

$$\begin{aligned} 12x-7y &< 120 \\ 7x-8y &\geq 36 \\ 5x+y &\geq 12 \end{aligned}$$

5.6 Linear Programming

Here you'll learn how to analyze and find the feasible solution(s) to a system of inequalities under a given set of constraints.

What if you had an equation like $z = x + y$ in which a set of contraints like $x - y \leq 4$, $x + y \leq 2$, and $2x + 3y \geq -3$ were placed on it. How could you find the minimum and maximum values of z? After completing this Concept, you'll be able to analyze a system of inequalities to make the best decisions given the constraints of the situation.

Guidance

A lot of interesting real-world problems can be solved with systems of linear inequalities.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend only waits tables in a certain region of the restaurant. The restaurant is also known for its great views, so you want to sit in a certain area of the restaurant that offers a good view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best view and be served by your friend.

Often, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the *best* possible value under those conditions. A typical example would be taking the limitations of materials and labor at a factory, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real-life systems can have dozens or hundreds of variables, or more. In this section, we'll only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the coordinate plane (called **the feasibility region**).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the system of equations that applies to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

Example A

If $z = 2x + 5y$, *find the maximum and minimum values of* z *given these constraints:*

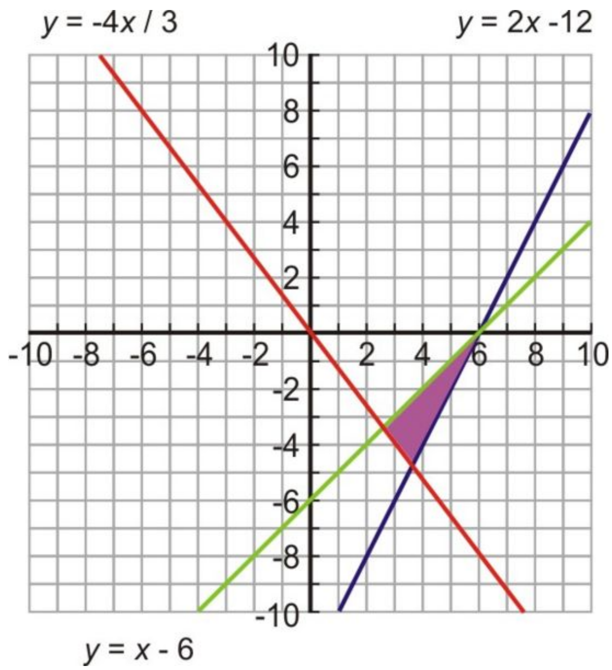
$$\begin{aligned} 2x - y &\leq 12 \\ 4x + 3y &\geq 0 \\ x - y &\geq 6 \end{aligned}$$

Solution

First, we need to find the solution to this system of linear inequalities by graphing and shading appropriately. To graph the inequalities, we rewrite them in slope-intercept form:

$$y \geq 2x - 12$$
$$y \geq -\frac{4}{3}x$$
$$y \leq x - 6$$

These three linear inequalities are called the **constraints**, and here is their graph:



The shaded region in the graph is called the **feasibility region**. All possible solutions to the system occur in that region; now we must try to find the maximum and minimum values of the variable z within that region. In other words, which values of x and y within the feasibility region will give us the greatest and smallest overall values for the expression $2x+5y$?

Fortunately, we don't have to test every point in the region to find that out. It just so happens that the minimum or maximum value of the optimization equation in a linear system like this will always be found at one of the vertices (the corners) of the feasibility region; we just have to figure out *which* vertices. So for each vertex—each point where two of the lines on the graph cross—we need to solve the system of just those two equations, and then find the value of z at that point.

The first system consists of the equations $y = 2x - 12$ and $y = -\frac{4}{3}x$. We can solve this system by substitution:

$$-\frac{4}{3}x = 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6$$
$$y = 2x - 12 \Rightarrow y = 2(3.6) - 12 \Rightarrow y = -4.8$$

The lines intersect at the point (3.6, -4.8).

The second system consists of the equations $y = 2x - 12$ and $y = x - 6$. Solving this system by substitution:

$$x - 6 = 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6$$
$$y = x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 0$$

The lines intersect at the point (6, 0).

The third system consists of the equations $y = -\frac{4}{3}x$ and $y = x - 6$. Solving this system by substitution:

$$x - 6 = -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57$$
$$y = x - 6 \Rightarrow y = 2.57 - 6 \Rightarrow y = -3.43$$

The lines intersect at the point (2.57, -3.43).

So now we have three different points that might give us the maximum and minimum values for z. To find out which ones actually do give the maximum and minimum values, we can plug the points into the optimization equation $z = 2x + 5y$.

When we plug in (3.6, -4.8), we get $z = 2(3.6) + 5(-4.8) = -16.8$.

When we plug in (6, 0), we get $z = 2(6) + 5(0) = 12$.

When we plug in (2.57, -3.43), we get $z = 2(2.57) + 5(-3.43) = -12.01$.

So we can see that **the point (6, 0) gives us the maximum possible value for z and the point (3.6, –4.8) gives us the minimum value.**

In the previous example, we learned how to apply the method of linear programming in the abstract. In the next example, we'll look at a real-life application.

Example B

You have $10,000 to invest, and three different funds to choose from. The municipal bond fund has a 5% return, the local bank's CDs have a 7% return, and a high-risk account has an expected 10% return. To minimize risk, you decide not to invest any more than $1,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. What's the best way to distribute your money given these constraints?

Solution:

Let's define our variables:

x is the amount of money invested in the municipal bond at 5% return

y is the amount of money invested in the bank's CD at 7% return

$10000 - x - y$ is the amount of money invested in the high-risk account at 10% return

z is the total interest returned from all the investments, so $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of z.

Now, let's write inequalities for the *constraints*:

You decide not to invest more than $1000 in the high-risk account—that means:

$$10000 - x - y \leq 1000$$

You need to invest at least three times as much in the municipal bonds as in the bank CDs—that means:

$$3y \leq x$$

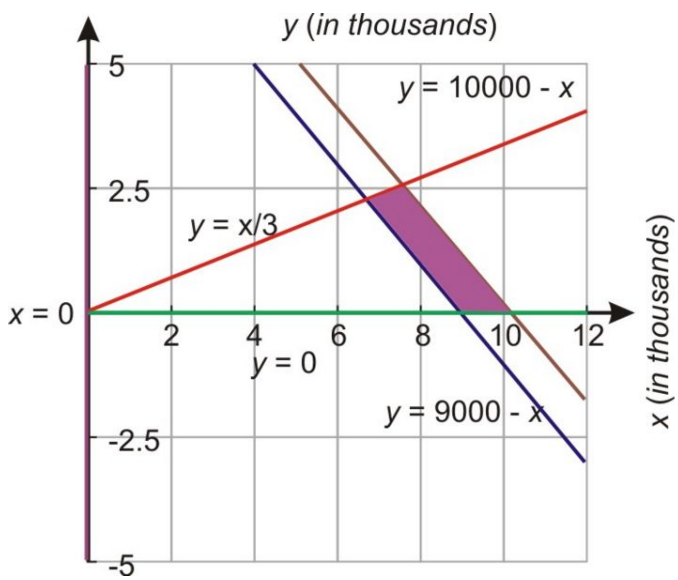
Also, you can't invest less than zero dollars in each account, so:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 10000 - x - y &\geq 0 \end{aligned}$$

To summarize, we must maximize the expression $z = 1000 - .05x - .03y$ using the constraints:

$$\begin{array}{lcl} 10000 - x - y \leq 1000 & & y \geq 9000 - x \\ 3y \leq x & & y \leq \frac{x}{3} \\ x \geq 0 & \text{Or in slope-intercept form:} & x \geq 0 \\ y \geq 0 & & y \geq 0 \\ 10000 - x - y \geq 0 & & y \leq 10000 - x \end{array}$$

Step 1: Find the solution region to the set of inequalities by graphing each line and shading appropriately. The following figure shows the overlapping region:



The purple region is the feasibility region where all the possible solutions can occur.

Step 2: Next we need to find the corner points of the feasibility region. Notice that there are four corners. To find their coordinates, we must pair up the relevant equations and solve each resulting system.

System 1:

$$\begin{aligned} y &= \frac{x}{3} \\ y &= 10000 - x \end{aligned}$$

Substitute the first equation into the second equation:

$$\frac{x}{3} = 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow 4x = 30000 \Rightarrow x = 7500$$
$$y = \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500$$

The intersection point is (7500, 2500).

System 2:

$$y = \frac{x}{3}$$
$$y = 9000 - x$$

Substitute the first equation into the second equation:

$$\frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750$$
$$y = \frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250$$

The intersection point is (6750, 2250).

System 3:

$$y = 0$$
$$y = 10000 - x$$

The intersection point is (10000, 0).

System 4:

$$y = 0$$
$$y = 9000 - x$$

The intersection point is (9000, 0).

Step 3: In order to find the maximum value for z, we need to plug all the intersection points into the equation for z and find which one yields the largest number.

(7500, 2500): $z = 1000 - 0.05(7500) - 0.03(2500) = 550$

(6750, 2250): $z = 1000 - 0.05(6750) - 0.03(2250) = 595$

(10000, 0): $z = 1000 - 0.05(10000) - 0.03(0) = 500$

(9000, 0): $z = 1000 - 0.05(9000) - 0.03(0) = 550$

The maximum return on the investment of \$595 occurs at the point (6750, 2250). This means that:

\$6,750 is invested in the municipal bonds.

\$2,250 is invested in the bank CDs.

\$1,000 is invested in the high-risk account.

Example C

James is trying to expand his pastry business to include cupcakes and personal cakes. He has 40 hours available to decorate the new items and can use no more than 22 pounds of cake mix. Each personal cake requires 2 pounds of cake mix and 2 hours to decorate. Each cupcake order requires one pound of cake mix and 4 hours to decorate. If he can sell each personal cake for \$14.99 and each cupcake order for \$16.99, how many personal cakes and cupcake orders should James make to make the most revenue?

There are four inequalities in this situation. First, state the variables. Let $p = \textit{the number of personal cakes}$ and $c = \textit{the number of cupcake orders}$.

Translate this into a system of inequalities.

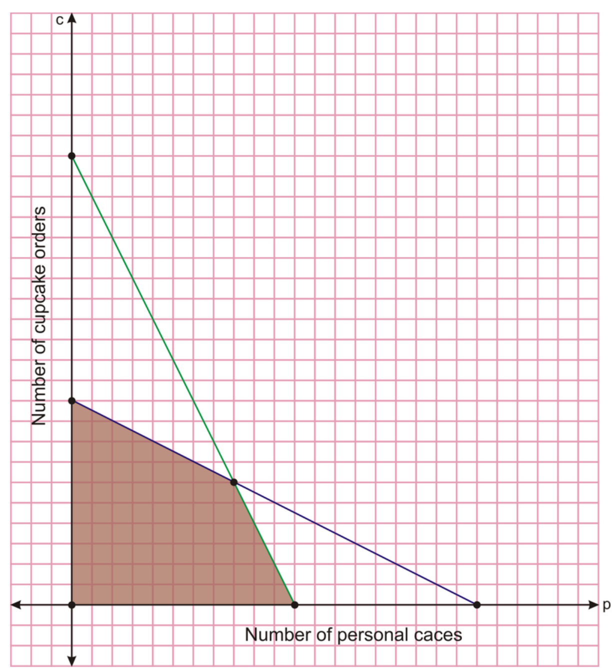
$2p + 1c \leq 22$ –This is the amount of available cake mix.

$2p + 4c \leq 40$ –This is the available time to decorate.

$p \geq 0$ –You cannot make negative personal cakes.

$c \geq 0$ –You cannot make negative cupcake orders.

Now graph each inequality and determine the feasible region.



The feasible region has four vertices: $\{(0,\ 0),(0,\ 10),(11,\ 0),(8,\ 6)\}$. According to our theorem, the optimization answer will **only occur** at one of these vertices.

Write the optimization equation. How much of each type of order should James make to bring in the most revenue?

$$14.99p + 16.99c = \textit{maximum revenue}$$

Substitute each ordered pair to determine which makes the most money.

$$\begin{aligned}(0,0) &\rightarrow \$0.00\\(0,10) &\rightarrow 14.99(0)+16.99(10)=\$169.90\\(11,0) &\rightarrow 14.99(11)+16.99(0)=\$164.89\\(8,6) &\rightarrow 14.99(8)+16.99(6)=\$221.86\end{aligned}$$

To make the most revenue, James should make 8 personal cakes and 6 cupcake orders.

Vocabulary

- **Linear programming** is the mathematical process of analyzing a system of inequalities to make the best decisions given the constraints of the situation.
- **Constraints** are the particular restrictions of a situation due to time, money, or materials.
- In an **optimization** problem, the goal is to locate the feasible region of the system and use it to answer a profitability, or ***optimization***, question.
- The **solution for the system of inequalities** is the common shaded region between all the inequalities in the system.
- The common shaded region of the system of inequalities is called the **feasible region**.

Guided Practice

Graph the solution to the following system:

$$\begin{aligned}x-y &< -6\\2y &\geq 3x+17\end{aligned}$$

Solution:

First we will rewrite the equations in slope-intercept form in order to graph them:

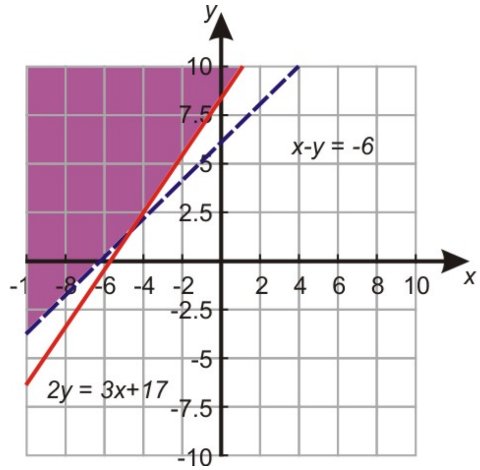
Inequality 1

$$\begin{aligned}x-y &< -6 && \text{Solve for y.}\\-y &< -x-6 && \text{Subtract x from each side.}\\y &> x+6 && \text{Multiply each side by -1, flipping the inequality.}\end{aligned}$$

Inequality 2

$$\begin{aligned}2y &\geq 3x+17< && \text{Solve for y.}\\y &\geq \frac{3}{2}x+8.5< && \text{Divide each side by 2.}\end{aligned}$$

Graph each equation and shade accordingly:



Practice

Solve the following linear programming problems.

1. Given the following constraints, find the maximum and minimum values of $z = -x + 5y$:

$$x + 3y \leq 0$$
$$x - y \geq 0$$
$$3x - 7y \leq 16$$

Santa Claus is assigning elves to work an eight-hour shift making toy trucks. Apprentice elves draw a wage of five candy canes per hour worked, but can only make four trucks an hour. Senior elves can make six trucks an hour and are paid eight candy canes per hour. There's only room for nine elves in the truck shop, and due to a candy-makers' strike, Santa Claus can only pay out 480 candy canes for the whole 8-hour shift.

2. How many senior elves and how many apprentice elves should work this shift to maximize the number of trucks that get made?
3. How many trucks will be made?
4. Just before the shift begins, the apprentice elves demand a wage increase; they insist on being paid seven candy canes an hour. Now how many apprentice elves and how many senior elves should Santa assign to this shift?
5. How many trucks will now get made, and how many candy canes will Santa have left over?

In Adrian's Furniture Shop, Adrian assembles both bookcases and TV cabinets. Each type of furniture takes her about the same time to assemble. She figures she has time to make at most 18 pieces of furniture by this Saturday. The materials for each bookcase cost her \$20 and the materials for each TV stand costs her \$45. She has \$600 to spend on materials. Adrian makes a profit of \$60 on each bookcase and a profit of \$100 on each TV stand.

6. Set up a system of inequalities. What $x-$ and $y-$values do you get for the point where Adrian's profit is maximized? Does this solution make sense in the real world?
7. What two possible real-world $x-$values and what two possible real-world $y-$values would be closest to the values in that solution?
8. With two choices each for x and y, there are four possible combinations of $x-$ and $y-$values. Of those four combinations, which ones actually fall within the feasibility region of the problem?
9. Which one of those feasible combinations seems like it would generate the most profit? Test out each one to confirm your guess. How much profit will Adrian make with that combination?

10. Based on Adrian's previous sales figures, she doesn't think she can sell more than 8 TV stands. Now how many of each piece of furniture should she make, and what will her profit be?
11. Suppose Adrian is confident she can sell all the furniture she can make, but she doesn't have room to display more than 7 bookcases in her shop. Now how many of each piece of furniture should she make, and what will her profit be?
12. Here's a "linear programming" problem on a line instead of a plane: Given the constraints $x \leq 5$ and $x \geq -2$, maximize the value of y where $y = x + 3$.

CHAPTER 6

Chapter 6: Exponential Functions

Chapter Outline

6.1 Product Rules for Exponents

Here you'll learn how to multiply two terms with the same base and how to find the power of a product.

Suppose you have the expression:

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$

How could you write this expression in a more concise way?

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and products.

RULE: To multiply two terms with the same base, add the exponents.

$$a^m \times a^n = \underbrace{(a \times a \times \ldots \times a)}_{\downarrow \atop m \text{ factors}} \underbrace{(a \times a \times \ldots \times a)}_{\downarrow \atop n \text{ factors}}$$

$$a^m \times a^n = \underbrace{(a \times a \times a \ldots \times a)}_{\downarrow \atop m+n \text{ factors}}$$

$$a^m \times a^n = a^{m+n}$$

RULE: To raise a product to a power, raise each of the factors to the power.

$$(ab)^n = \underbrace{(ab) \times (ab) \times \ldots \times (ab)}_{\downarrow \atop n \text{ factors}}$$

$$(ab)^n = \underbrace{(a \times a \times \ldots \times a)}_{\downarrow \atop n \text{ factors}} \times \underbrace{(b \times b \times \ldots \times b)}_{\downarrow \atop n \text{ factors}}$$

$$(ab)^n = a^n b^n$$

Example A

Simplify $3^2 \times 3^3$.

Solution:

$3^2 \times 3^3$	The base is 3.
3^{2+3}	Keep the base of 3 and add the exponents.
3^5	This answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3$$
$$3^5 = 243$$
$$\boxed{3^2 \times 3^3 = 3^5 = 243}$$

Example B

Simplify $(x^3)(x^6)$.

Solution:

$(x^3)(x^6)$	The base is x.
x^{3+6}	Keep the base of x and add the exponents.
x^9	The answer is in exponential form.
$\boxed{(x^3)(x^6) = x^9}$	

Example C

Simplify $y^5 \cdot y^2$.

Solution:

$y^5 \cdot y^2$	The base is y.
y^{5+2}	Keep the base of y and add the exponents.
y^7	The answer is in exponential form.
$\boxed{y^5 \cdot y^2 = y^7}$	

Example D

Simplify $5x^2y^3 \cdot 3xy^2$.

Solution:

$5x^2y^3 \cdot 3xy^2$	The bases are x and y.
$15(x^2y^3)(xy^2)$	Multiply the coefficients - $5 \times 3 = 15$. Keep the base of x and y and add the exponents of the same base. If a base does not have a written exponent, it is understood as 1.
$15x^{2+1}y^{3+2}$	
$15x^3y^5$	The answer is in exponential form.
$\boxed{5x^2y^3 \cdot 3xy^2 = 15x^3y^5}$	

Concept Problem Revisited

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$ can be rewritten as $x^9y^5x^4$. Then, you can use the rules of exponents to simplify the expression to $x^{13}y^5$. This is certainly much quicker to write!

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

Simplify each of the following expressions.

1. $(-3x)^2$

2. $(5xy)^3$

3. $(2^3 \cdot 3^2)^2$

Answers:

1. $9x^2$. Here are the steps:

$$\begin{aligned}(-3x)^2 &= (-3)^2 \cdot (x)^2 \\ &= 9x^2\end{aligned}$$

2. $125x^3y^3$. Here are the steps:

$$\begin{aligned}(5x^2y^4)^3 &= (5)^3 \cdot (x)^3 \cdot (y)^3\\ &= 125x^3y^3\end{aligned}$$

3. 5184. Here are the steps:

$$\begin{aligned}(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2\\ &= (72)^2\\ &= 5184\end{aligned}$$

OR

$$\begin{aligned}(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2\\ &= 8^2 \cdot 9^2\\ &= 64 \cdot 81\\ &= 5184\end{aligned}$$

Practice

Simplify each of the following expressions, if possible.

1. $4^2 \times 4^4$
2. $x^4 \cdot x^{12}$
3. $(3x^2y^4)(9xy^5z)$
4. $(2xy)^2(4x^2y^3)$
5. $(3x)^5(2x)^2(3x^4)$
6. $x^3y^2z \cdot 4xy^2z^7$
7. $x^2y^3 + xy^2$
8. $(0.1xy)^4$
9. $(xyz)^6$
10. $2x^4(x^2 - y^2)$
11. $3x^5 - x^2$
12. $3x^8(x^2 - y^4)$

Expand and then simplify each of the following expressions.

13. $(x^5)^3$
14. $(x^6)^8$
15. $(x^a)^b$ *Hint: Look for a pattern in the previous two problems.*

6.2 Quotient Rules for Exponents

Here you'll learn how to divide two terms with the same base and find the power of a quotient.

Suppose you have the expression:

$$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}$$

How could you write this expression in a more concise way?

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and quotients.

RULE: To divide two powers with the same base, subtract the exponents.

$$\frac{a^m}{a^n} = \frac{\overbrace{(a \times a \times \ldots \times a)}^{m \text{ factors}}}{\underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}}} \quad m > n; a \neq 0$$

$$\frac{a^m}{a^n} = \underbrace{(a \times a \times \ldots \times a)}_{m-n \text{ factors}}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

RULE: To raise a quotient to a power, raise both the numerator and the denominator to the power.

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b}}$$

$\downarrow$

n factors

n factors

$\uparrow$

$$\left(\frac{a}{b}\right)^n = \frac{\overbrace{(a \times a \times \ldots \times a)}}{\underbrace{(b \times b \times \ldots \times b)}}$$

$\downarrow$

n factors

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\ (b \neq 0)$$

Example A

Simplify $2^7 \div 2^3$.

Solution:

$2^7 \div 2^3$	The base is 2.
2^{7-3}	Keep the base of 2 and subtract the exponents.
2^4	The answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$2^4 = 2 \times 2 \times 2 \times 2$$
$$2^4 = 16$$
$$\boxed{2^7 \div 2^3 = 2^4 = 16}$$

Example B

Simplify $\frac{x^8}{x^2}$.

Solution:

$\frac{x^8}{x^2}$	The base is x.
x^{8-2}	Keep the base of x and subtract the exponents.
x^6	The answer is in exponential form.

$$\boxed{\frac{x^8}{x^2} = x^6}$$

Example C

Simplify $\frac{16x^5y^5}{4x^2y^3}$.

Solution:

$\frac{16x^5y^5}{4x^2y^3}$	The bases are x and y.
$4\left(\frac{x^5y^5}{x^2y^3}\right)$	Divide the coefficients - $16 \div 4 = 4$. Keep the base of x and y and subtract the exponents of the same base.
$4x^{5-2}y^{5-3}$	
$4x^3y^2$	

Concept Problem Revisited

$\frac{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y\cdot y\cdot y}{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y}$ can be rewritten as $\frac{x^9y^5}{x^6y^3}$ and then simplified to x^3y^2.

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

Simplify each of the following expressions.

1. $\left(\frac{2}{3}\right)^2$

2. $\left(\frac{x}{6}\right)^3$

3. $\left(\frac{3x}{4y}\right)^2$

Answers:

1. $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$
2. $\left(\frac{x}{6}\right)^3 = \frac{x^3}{6^3} = \frac{x^3}{216}$
3. $\left(\frac{3x}{4y}\right)^2 = \frac{3^2x^2}{4^2y^2} = \frac{9x^2}{16y^2}$

Practice

Simplify each of the following expressions, if possible.

1. $\left(\frac{2}{5}\right)^6$
2. $\left(\frac{4}{7}\right)^3$
3. $\left(\frac{x}{y}\right)^4$
4. $\frac{20x^4y^5}{5x^2y^4}$
5. $\frac{42x^2y^8z^2}{6xy^4z}$
6. $\left(\frac{3x}{4y}\right)^3$
7. $\frac{72x^2y^4}{8x^2y^3}$
8. $\left(\frac{x}{4}\right)^5$
9. $\frac{24x^{14}y^8}{3x^5y^7}$
10. $\frac{72x^3y^9}{24xy^6}$
11. $\left(\frac{7}{y}\right)^3$
12. $\frac{20x^{12}}{-5x^8}$

13. Simplify using the laws of exponents: $\frac{2^3}{2^5}$
14. Evaluate the numerator and denominator separately and then simplify the fraction: $\frac{2^3}{2^5}$
15. Use your result from the previous problem to determine the value of a: $\frac{2^3}{2^5} = \frac{1}{2^a}$
16. Use your results from the previous three problems to help you evaluate 2^{-4}.

6.3 Power Rule for Exponents

Here you'll learn how to find the power of a power.

Can you simplify an expression where an exponent has an exponent? For example, how would you simplify $[(2^3)^2]^4$?

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn a rule that has to do with raising a power to another power.

RULE: To raise a power to a new power, multiply the exponents.

$$(a^m)^n = \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}{}^n$$

$$(a^m)^n = \underbrace{\underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \times \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}}_{n \text{ times}}$$

$$(a^m)^n = \underbrace{a \times a \times a \ldots \times a}_{mn \text{ factors}}$$

$$(a^m)^n = a^{mn}$$

Example A

Evaluate $(2^3)^2$.

Solution: $(2^3)^2 = 2^6 = 64$.

Example B

Simplify $(x^7)^4$.

Solution: $(x^7)^4 = x^{28}$.

Example C

Evaluate $(3^2)^3$.

Solution: $(3^2)^3 = 3^6 = 729$.

Example D

Simplify $(x^2y^4)^2 \cdot (xy^4)^3$.

Solution: $(x^2y^4)^2 \cdot (xy^4)^3 = x^4y^8 \cdot x^3y^{12} = x^7y^{20}$.

Concept Problem Revisited

$[(2^3)^2]^4 = [2^6]^4 = 2^{24}$. Notice that the power rule applies even when a number has been raised to more than one power. The overall exponent is 24 which is $3 \cdot 2 \cdot 4$.

Vocabulary

Base
In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent
In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.
In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents
The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

You know you can rewrite 2^4 as $2 \times 2 \times 2 \times 2$ and then calculate in order to find that

$$\boxed{2^4 = 16}$$

. This concept can also be reversed. To write 32 as a power of 2, $32 = 2 \times 2 \times 2 \times 2 \times 2$. There are 5 twos; therefore,

$$\boxed{32 = 2^5}$$

. Use this idea to complete the following problems.

1. Write 81 as a power of 3.

2. Write $(9)^3$ as a power of 3.

3. Write $(4^3)^2$ as a power of 2.

Answers:

1. $81 = 3 \times 3 = 9 \times 3 = 27 \times 3 = 81$

There are 4 threes. Therefore

$$\boxed{81 = 3^4}$$

2. $9 = 3 \times 3 = 9$

There are 2 threes. Therefore

$$\boxed{9 = 3^2}$$

.

$(3^2)^3$ Apply the law of exponents for power to a power-multiply the exponents.
$3^{2\times 3} = 3^6$
Therefore

$$\boxed{(9)^3 = 3^6}$$

3. $4 = 2 \times 2 = 4$

There are 2 twos. Therefore

$$\boxed{4 = 2^2}$$

$\left((2^2)^3\right)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{2\times 3} = 2^6}$$

$(2^6)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{6\times 2} = 2^{12}}$$

Therefore

$$\boxed{(4^3)^2 = 2^{12}}$$

Practice

Simplify each of the following expressions.

1. $\left(\frac{x^4}{y^3}\right)^5$
2. $\frac{(5x^2y^4)^5}{(5xy^2)^3}$
3. $\frac{x^8y^9}{(x^2y)^3}$
4. $(x^2y^4)^3$
5. $(3x^2)^2 \cdot (4xy^4)^2$
6. $(2x^3y^5)(5x^2y)^3$

7. $(x^4y^6z^2)^2(3xyz)^3$
8. $\left(\frac{x^2}{2y^3}\right)^4$
9. $\frac{(4xy^3)^4}{(2xy^2)^3}$
10. True or false: $(x^2+y^3)^2 = x^4+y^6$
11. True or false: $(x^2y^3)^2 = x^4y^6$
12. Write 64 as a power of 4.
13. Write $(16)^3$ as a power of 2.
14. Write $(9^4)^2$ as a power of 3.
15. Write $(81)^2$ as a power of 3.
16. Write $(25^3)^4$ as a power of 5.

6.4 Exponential Growth Function

Here you'll learn how to analyze an exponential growth function and its graph.

A population of 10 mice grows at a rate of 300% every month. How many mice are in the population after six months?

Guidance

An **exponential function** has the variable in the exponent of the expression. All exponential functions have the form: $f(x) = a \cdot b^{x-h} + k$, where h and k move the function in the x and y directions respectively, much like the other functions we have seen in this text. b is the base and a changes how quickly or slowly the function grows. Let's take a look at the parent graph, $y = 2^x$.

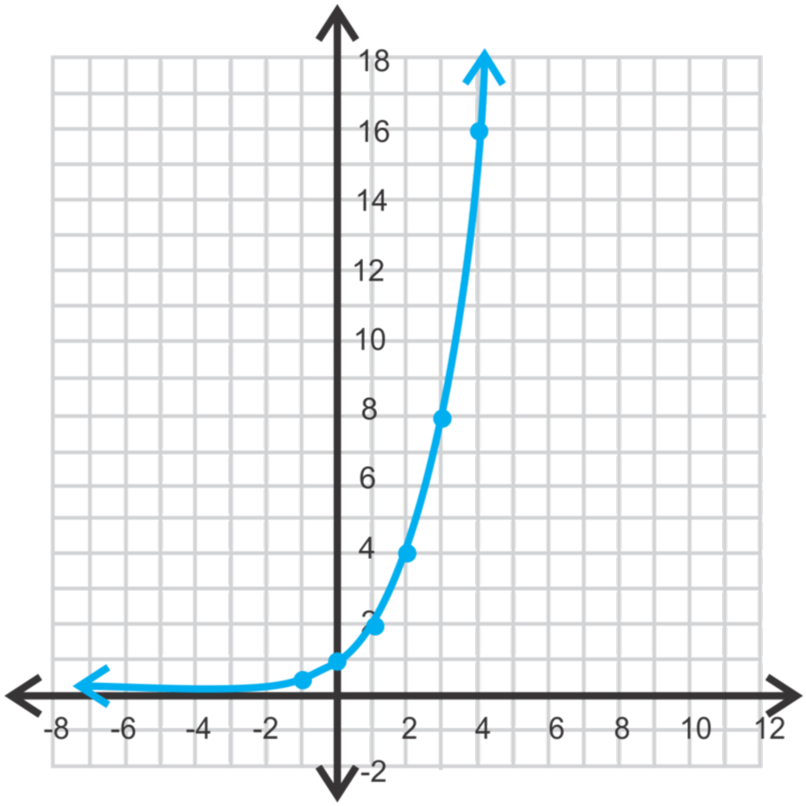
Example A

Graph $y = 2^x$. Find the y-intercept.

Solution: Let's start by making a table. Include some positive and negative values for x and zero.

TABLE 6.1:

x	2^x	y
3	2^3	8
2	2^2	4
1	2^1	2
0	2^0	1
-1	2^{-1}	$\frac{1}{2}$
-2	2^{-2}	$\frac{1}{4}$
-3	2^{-3}	$\frac{1}{8}$



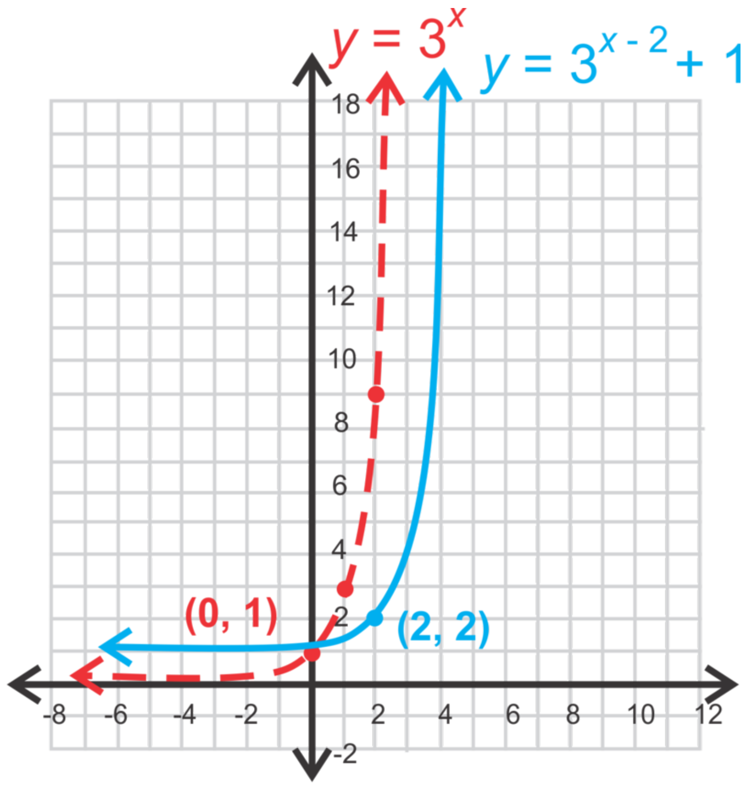
This is the typical shape of an **exponential growth function**. The function grows "exponentially fast". Meaning, in this case, the function grows in powers of 2. For an exponential function to be a growth function, $a > 0$ and $b > 1$ and h and k are both zero $(y = ab^x)$. From the table, we see that the y-intercept is (0, 1).

Notice that the function gets very, very close to the x-axis, but never touches or passes through it. Even if we chose $x = -50,\ y$ would be $2^{-50} = \frac{1}{2^{50}}$, which is still not zero, but very close. In fact, the function will never reach zero, even though it will get smaller and smaller. Therefore, this function approaches the line $y = 0$, but will never touch or pass through it. This type of boundary line is called an **asymptote**. In the case with all exponential functions, there will be a horizontal asymptote. If $k = 0$, then the asymptote will be $y = 0$.

Example B

Graph $y = 3^{x-2} + 1$. Find the y-intercept, asymptote, domain and range.

Solution: This is not considered a growth function because h and k are not zero. To graph something like this (without a calculator), start by graphing $y = 3^x$ and then shift it h units in the x-direction and k units in the y-direction.



Notice that the point (0, 1) from $y = 3^x$ gets shifted to the right 2 units and up one unit and is (2, 2) in the translated function, $y = 3^{x-2} + 1$. Therefore, the asymptote is $y = 1$. To find the y-intercept, plug in $x = 0$.

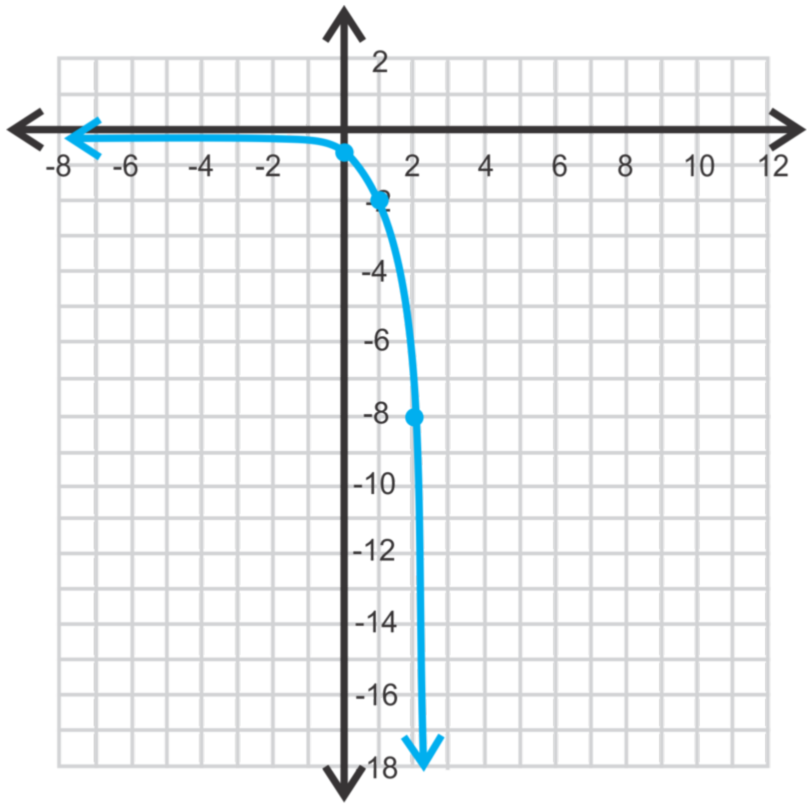
$$y = 3^{0-2} + 1 = 3^{-2} + 1 = 1\frac{1}{9} = 1.\overline{1}$$

The domain of all exponential functions is all real numbers. The range will be everything greater than the asymptote. In this example, the range is $y > 1$.

Example C

Graph the function $y = -\frac{1}{2} \cdot 4^x$. Determine if it is an exponential growth function.

Solution: In this example, we will outline how to use the graphing calculator to graph an exponential function. First, clear out anything in Y=. Next, input the function into Y1= -(1/2)4^X and press **GRAPH**. Adjust your window accordingly.



This is not an exponential growth function, because it does not grow in a positive direction. By looking at the definition of a growth function, $a > 0$, and it is not here.

Intro Problem Revisit

This is an example of exponential growth, so we can use the exponential form $f(x) = a \cdot b^{x-h} + k$. In this case, a = 10, the starting population; b = 300% or 3, the rate of growth; x-h = 6 the number of months, and k = 0.

$$\begin{aligned} P &= 10 \cdot 3^6 \\ &= 10 \cdot 729 = 7290 \end{aligned}$$

Therefore, the mouse population after six months is 7,290.

Guided Practice

Graph the following exponential functions. Determine if they are growth functions. Then, find the y-intercept, asymptote, domain and range. Use an appropriate window.

1. $y = 3^{x-4} - 2$

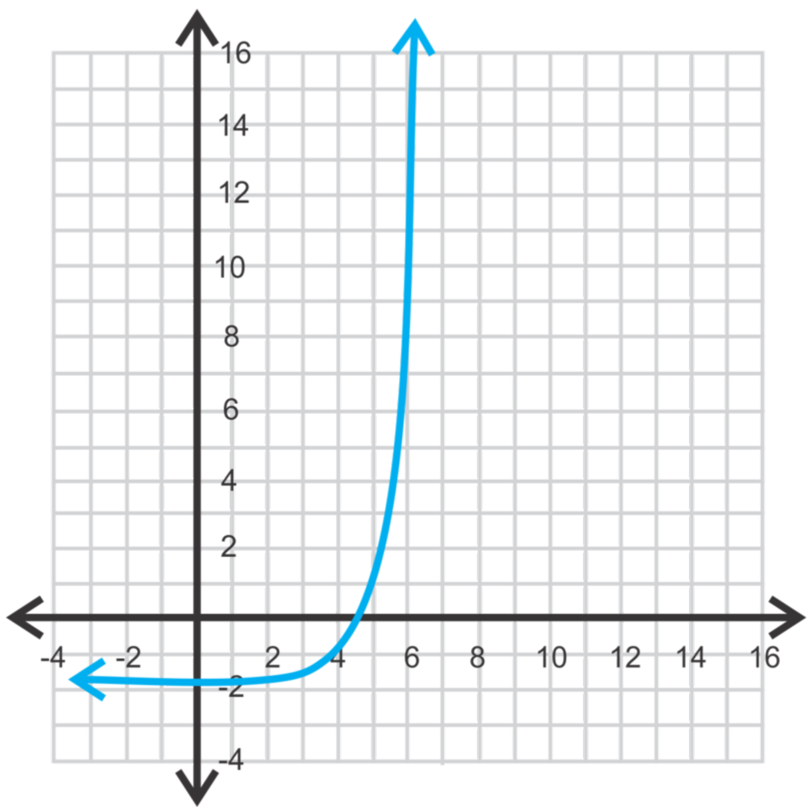
2. $f(x) = (-2)^{x+5}$

3. $f(x) = 5^x$

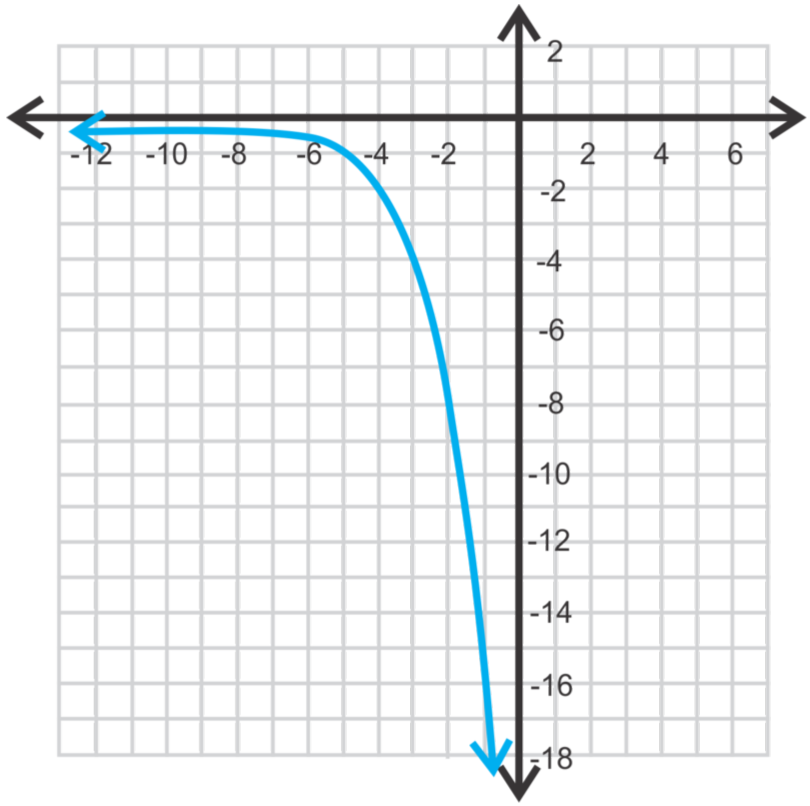
4. Abigail is in a singles tennis tournament. She finds out that there are eight rounds until the final match. If the tournament is single elimination, how many games will be played? How many competitors are in the tournament?

Answers

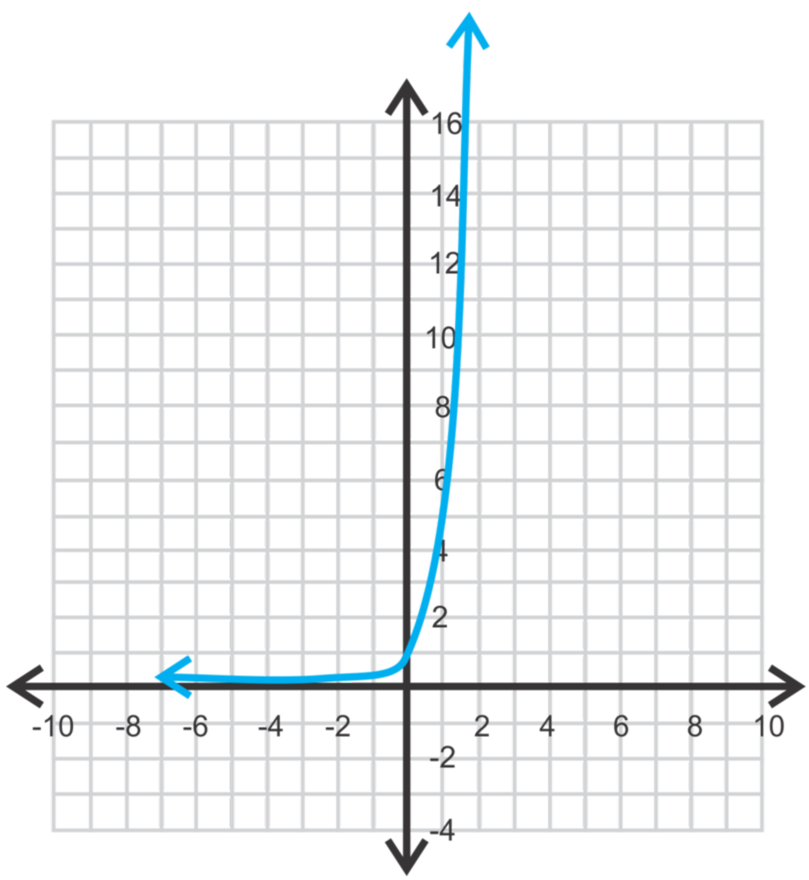
1. This is not a growth function because h and k are not zero. The y-intercept is $y = 3^{0-4} - 2 = \frac{1}{81} - 2 = -1\frac{80}{81}$, the asymptote is at $y = -2$, the domain is all real numbers and the range is $y > -2$.



2. This is not a growth function because h is not zero. The y-intercept is $y = (-2)^{0+5} = (-2)^5 = -32$, the asymptote is at $y = 0$, the domain is all real numbers and the range is $y > 0$.



3. This is a growth function. The y-intercept is $y = 5^{\circ} = 1$, the asymptote is at $y = 0$, the domain is all real numbers and the range is $y > 0$.



4. If there are eight rounds to single's games, there are will be $2^8 = 256$ competitors. In the first round, there will be 128 matches, then 64 matches, followed by 32 matches, then 16 matches, 8, 4, 2, and finally the championship game. Adding all these all together, there will be $128+64+32+16+8+4+2+1$ or 255 total matches.

Vocabulary

Exponential Function
A function whose variable is in the exponent. The general form is $y = a \cdot b^{x-h} + k$.

Exponential Growth Function
A specific type of exponential function where $h = k = 0, a > 0$, and $b > 1$. The general form is $y = ab^x$.

Asymptote
A boundary line that restricts the domain or range. This line is not part of the graph.

Practice

Graph the following exponential functions. Find the y-intercept, the equation of the asymptote and the domain and range for each function.

1. $y = 4^x$
2. $y = (-1)(5)^x$
3. $y = 3^x - 2$
4. $y = 2^x + 1$
5. $y = 6^{x+3}$
6. $y = -\frac{1}{4}(2)^x + 3$
7. $y = 7^{x+3} - 5$
8. $y = -(3)^{x-4} + 2$
9. $y = 3(2)^{x+1} - 5$
10. What is the y-intercept of $y = a^x$? Why is that?
11. What is the range of the function $y = a^{x-h} + k$?
12. March Madness is a single-game elimination tournament of 64 college basketball teams. How many games will be played until there is a champion? Include the championship game.
13. In 2012, the tournament added 4 teams to make it a field of 68 and there are 4 "play-in" games at the beginning of the tournament. How many games are played now?
14. An investment grows according the function $A = P(1.05)^t$ where P represents the initial investment, A represents the value of the investment and t represents the number of years of investment. If $10,000 was the initial investment, how much would the value of the investment be after 10 years, to the nearest dollar?
15. How much would the value of the investment be after 20 years, to the nearest dollar?

CHAPTER 7

Chapter 7: Polynomials

Chapter Outline

7.1 Polynomials in Standard Form

Here you'll learn how to identify polynomials and find their degree. You'll also learn how to write polynomial expressions in standard form and simplify them by combining like terms.

What if you were given an algebraic expression like $3x - 2x^2 + 5 - x + 6x^2$? How could you simplify it and find its degree? After completing this Concept, you'll be able to combine like terms to simplify polynomial expressions like this one and classify them by degree.

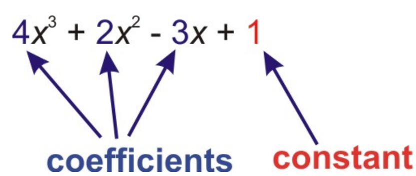
Guidance

So far we've seen functions described by straight lines (linear functions) and functions where the variable appeared in the exponent (exponential functions). In this section we'll introduce polynomial functions. A **polynomial** is made up of different terms that contain **positive integer** powers of the variables. Here is an example of a polynomial:

$$4x^3 + 2x^2 - 3x + 1$$

Each part of the polynomial that is added or subtracted is called a **term** of the polynomial. The example above is a polynomial with *four terms*.

The numbers appearing in each term in front of the variable are called the **coefficients**. The number appearing all by itself without a variable is called a **constant**.



In this case the coefficient of x^3 is **4**, the coefficient of x^2 is **2**, the coefficient of x is **-3** and the constant is **1**.

Degrees of Polynomials and Standard Form

Each term in the polynomial has a different **degree**. The degree of the term is the power of the variable in that term.

$4x^3$	has degree 3 and is called a cubic term or 3^{rd} order term.
$2x^2$	has degree 2 and is called a quadratic term or 2^{nd} order term.
$-3x$	has degree 1 and is called a linear term or 1^{st} order term.
1	has degree 0 and is called the constant.

By definition, **the degree of the polynomial** is the same as the degree of the term with the highest degree. This example is a polynomial of degree 3, which is also called a "cubic" polynomial. (Why do you think it is called a cubic?).

Polynomials can have more than one variable. Here is another example of a polynomial:

$$t^4 - 6s^3t^2 - 12st + 4s^4 - 5$$

This is a polynomial because all the exponents on the variables are positive integers. This polynomial has five terms. Let's look at each term more closely.

Note: *The degree of a term is the sum of the powers on each variable in the term.* In other words, the degree of each term is the number of variables that are multiplied together in that term, whether those variables are the same or different.

$$\begin{array}{ll} t^4 & \text{has a degree of } 4, \text{ so it's a } 4^{th} \text{ order term} \\ -6s^3t^2 & \text{has a degree of } 5, \text{ so it's a } 5^{th} \text{ order term.} \\ -12st & \text{has a degree of } 2, \text{ so it's a } 2^{nd} \text{ order term.} \\ 4s^4 & \text{has a degree of } 4, \text{ so it's a } 4^{th} \text{ order term.} \\ -5 & \text{is a constant, so its degree is } 0. \end{array}$$

Since the highest degree of a term in this polynomial is 5, then this is polynomial of degree 5^{th} or a 5^{th} order polynomial.

A polynomial that has only one term has a special name. It is called a **monomial** (*mono* means one). A monomial can be a constant, a variable, or a product of a constant and one or more variables. You can see that each term in a polynomial is a monomial, so a polynomial is just the sum of several monomials. Here are some examples of monomials:

$$b^2 \qquad -2ab^2 \qquad 8 \qquad \frac{1}{4}x^4 \qquad -29xy$$

Example A

For the following polynomials, identify the coefficient of each term, the constant, the degree of each term and the degree of the polynomial.

a) $x^5-3x^3+4x^2-5x+7$

b) $x^4-3x^3y^2+8x-12$

Solution

a) $x^5-3x^3+4x^2-5x+7$

The coefficients of each term in order are 1, -3, 4, and -5 and the constant is 7.

The degrees of each term are 5, 3, 2, 1, and 0. Therefore the degree of the polynomial is 5.

b) $x^4-3x^3y^2+8x-12$

The coefficients of each term in order are 1, -3, and 8 and the constant is -12.

The degrees of each term are 4, 5, 1, and 0. Therefore the degree of the polynomial is 5.

Example B

Identify the following expressions as polynomials or non-polynomials.

a) $5x^5-2x$

b) $3x^2-2x^{-2}$

c) $x\sqrt{x}-1$

d) $\frac{5}{x^3+1}$

e) $4x^{\frac{1}{3}}$

f) $4xy^2 - 2x^2y - 3 + y^3 - 3x^3$

Solution

a) This ***is*** a polynomial.

b) This is ***not*** a polynomial because it has a negative exponent.

c) This is ***not*** a polynomial because it has a radical.

d) This is ***not*** a polynomial because the power of x appears in the denominator of a fraction (and there is no way to rewrite it so that it does not).

e) This is ***not*** a polynomial because it has a fractional exponent.

f) This ***is*** a polynomial.

Often, we arrange the terms in a polynomial in order of decreasing power. This is called **standard form**.

The following polynomials are in standard form:

$$4x^4 - 3x^3 + 2x^2 - x + 1$$

$$a^4b^3 - 2a^3b^3 + 3a^4b - 5ab^2 + 2$$

The first term of a polynomial in standard form is called the **leading term**, and the coefficient of the leading term is called the **leading coefficient**.

The first polynomial above has the leading term $4x^4$, and the leading coefficient is 4.

The second polynomial above has the leading term a^4b^3, and the leading coefficient is 1.

Example C

Rearrange the terms in the following polynomials so that they are in standard form. Indicate the leading term and leading coefficient of each polynomial.

a) $7 - 3x^3 + 4x$

b) $ab - a^3 + 2b$

c) $-4b + 4 + b^2$

Solution

a) $7 - 3x^3 + 4x$ becomes $-3x^3 + 4x + 7$. Leading term is $-3x^3$; leading coefficient is -3.

b) $ab - a^3 + 2b$ becomes $-a^3 + ab + 2b$. Leading term is $-a^3$; leading coefficient is -1.

c) $-4b + 4 + b^2$ becomes $b^2 - 4b + 4$. Leading term is b^2; leading coefficient is 1.

Simplifying Polynomials

A polynomial is simplified if it has no terms that are alike. **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, whether they have the same or different coefficients.

For example, $2x^2y$ and $5x^2y$ are like terms, but $6x^2y$ and $6xy^2$ are not like terms.

When a polynomial has like terms, we can simplify it by combining those terms.

$$x^2 + \underline{6xy} - \underline{4xy} + y^2$$

Like terms

We can simplify this polynomial by combining the like terms $6xy$ and $-4xy$ into $(6-4)xy$, or $2xy$. The new polynomial is $x^2 + 2xy + y^2$.

Example D

Simplify the following polynomials by collecting like terms and combining them.

a) $2x - 4x^2 + 6 + x^2 - 4 + 4x$

b) $a^3b^3 - 5ab^4 + 2a^3b - a^3b^3 + 3ab^4 - a^2b$

Solution

a) Rearrange the terms so that like terms are grouped together: $(-4x^2 + x^2) + (2x + 4x) + (6 - 4)$

Combine each set of like terms: $-3x^2 + 6x + 2$

b) Rearrange the terms so that like terms are grouped together: $(a^3b^3 - a^3b^3) + (-5ab^4 + 3ab^4) + 2a^3b - a^2b$

Combine each set of like terms: $0 - 2ab^4 + 2a^3b - a^2b = -2ab^4 + 2a^3b - a^2b$

Vocabulary

- A ***polynomial*** is an expression made with constants, variables, and *positive integer* exponents of the variables.
- In a polynomial, the number appearing in each term in front of the variables is called the **coefficient.**
- In a polynomial, the number appearing all by itself without a variable is called the **constant.**
- A **monomial** is a one-termed polynomial. It can be a constant, a variable, or a variable with a coefficient.
- The **degree of a polynomial** is the largest degree of the terms. The **degree of a term** is the power of the variable, or if the term has more than one variable, it is the sum of the powers on each variable.
- We arrange the terms in a polynomial in **standard form** in which the term with the highest degree is first and is followed by the other terms in order of decreasing powers.
- **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, but they can have different coefficients.

Guided Practice

Simplify and rewrite the following polynomial in standard form. State the degree of the polynomial.

$16x^2y^3 - 3xy^5 - 2x^3y^2 + 2xy - 7x^2y^3 + 2x^3y^2$

Solution:

Start by simplifying by combining like terms:

$16x^2y^3 - 3xy^5 - 2x^3y^2 + 2xy - 7x^2y^3 + 2x^3y^2$

is equal to

$(16x^2y^3 - 7x^2y^3) - 3xy^5 + (-2x^3y^2 + 2x^3y^2) + 2xy$

which simplifies to

$9x^2y^3 - 3xy^5 + 2xy.$

In order to rewrite in standard form, we need to determine the degree of each term. The first term has a degree of $2+3=5$, the second term has a degree of $1+5=6$, and the last term has a degree of $1+1=2$. We will rewrite the terms in order from largest degree to smallest degree:

$-3xy^5 + 9x^2y^3 + 2xy$

The degree of a polynomial is the largest degree of all the terms. In this case that is 6.

Practice

Indicate whether each expression is a polynomial.

1. $x^2 + 3x^{\frac{1}{2}}$
2. $\frac{1}{3}x^2y - 9y^2$
3. $3x^{-3}$
4. $\frac{2}{3}t^2 - \frac{1}{t^2}$
5. $\sqrt{x} - 2x$
6. $\left(x^{\frac{3}{2}}\right)^2$

Express each polynomial in standard form. Give the degree of each polynomial.

7. $3 - 2x$
8. $8 - 4x + 3x^3$
9. $-5 + 2x - 5x^2 + 8x^3$
10. $x^2 - 9x^4 + 12$
11. $5x + 2x^2 - 3x$

7.2 Adding and Subtracting Polynomials

Here you'll learn how to add and subtract polynomials, as well as learn about the different parts of a polynomial.

Rectangular prism A has a volume of x^3+2x^2-3. Rectangular prism B has a volume of $x^4+2x^3-8x^2$. How much larger is the volume of rectangular prism B than rectangular prism A?

Guidance

A **polynomial** is an expression with multiple variable terms, such that the exponents are greater than or equal to zero. All quadratic and linear equations are polynomials. Equations with negative exponents, square roots, or variables in the denominator are not polynomials.

Polynomials	Not Polynomials
$2x^2+6x-9$	$10x^{-1}+6x^2$
$-x^3+9$	$\sqrt{x}-2$
$4x^4+5x^3-8x^2+12x+24$	$\frac{3}{x}+5$

Now that we have established what a polynomial is, there are a few important parts. Just like with a quadratic, a polynomial can have a **constant**, which is a number without a variable. The **degree** of a polynomial is the largest exponent. For example, all quadratic equations have a degree of 2. Lastly, the **leading coefficient** is the coefficient in front of the variable with the degree. In the polynomial $4x^4+5x^3-8x^2+12x+24$ above, the degree is 4 and the leading coefficient is also 4. Make sure that when finding the degree and leading coefficient you have the polynomial in standard form. **Standard form** lists all the variables in order, from greatest to least.

Example A

Rewrite $x^3-5x^2+12x^4+15-8x$ in standard form and find the degree and leading coefficient.

Solution: To rewrite in standard form, put each term in order, from greatest to least, according to the exponent. Always write the constant last.

$$x^3-5x^2+12x^4+15-8x \rightarrow 12x^4+x^3-5x^2-8x+15$$

Now, it is easy to see the leading coefficient, 12, and the degree, 4.

Example B

Simplify $(4x^3-2x^2+4x+15)+(x^4-8x^3-9x-6)$

Solution: To add or subtract two polynomials, combine like terms. **Like terms** are any terms where the exponents of the variable are the same. We will regroup the polynomial to show the like terms.

$$\begin{aligned}&(4x^3-2x^2+4x+15)+(x^4-8x^3-9x-6)\\&x^4+(4x^3-8x^3)-2x^2+(4x-9x)+(15-6)\\&x^4-4x^3-2x^2-5x+9\end{aligned}$$

Example C

Simplify $(2x^3+x^2-6x-7)-(5x^3-3x^2+10x-12)$

Solution: When subtracting, distribute the negative sign to every term in the second polynomial, then combine like terms.

$$\begin{aligned}&(2x^3+x^2-6x-7)-(5x^3-3x^2+10x-12)\\&2x^3+x^2-6x-7-5x^3+3x^2-10x+12\\&(2x^3-5x^3)+(x^2+3x^2)+(-6x-10x)+(-7+12)\\&-3x^3+4x^2-16x+5\end{aligned}$$

Intro Problem Revisit

We need to subtract the volume of rectangular prism A from the volume of rectangular prism B.

$$\begin{aligned}(x^4+2x^3-8x^2)-(x^3+2x^2-3)\\=x^4+2x^3-8x^2-x^3-2x^2+3\\=x^4+x^3-10x^2+3\end{aligned}$$

.

Therefore, the difference between the two volumes is $x^4+x^3-10x^2+3$.

Guided Practice

1. Is $\sqrt{2x^3-5x}+6$ a polynomial? Why or why not?

2. Find the leading coefficient and degree of $6x^2-3x^5+16x^4+10x-24$.

Add or subtract.

3. $(9x^2+4x^3-15x+22)+(6x^3-4x^2+8x-14)$

4. $(7x^3+20x-3)-(x^3-2x^2+14x-18)$

Answers

1. No, this is not a polynomial because x is under a square root in the equation.

2. In standard form, this polynomial is $-3x^5+16x^4+6x^2+10x-24$. Therefore, the degree is 5 and the leading coefficient is -3.

3. $(9x^2+4x^3-15x+22)+(6x^3-4x^2+8x-14)=10x^3+5x^2-7x+8$

4. $(7x^3+20x-3)-(x^3-2x^2+14x-18)=6x^3+2x^2+6x+15$

Vocabulary

Polynomial
An expression with multiple variable terms, such that the exponents are greater than or equal to zero.

Constant
A number without a variable in a mathematical expression.

Degree(of a polynomial)
The largest exponent in a polynomial.

Leading coefficient
The coefficient in front of the variable with the degree.

Standard form
Lists all the variables in order, from greatest to least.

Like terms
Any terms where the exponents of the variable are the same.

Practice

Determine if the following expressions are polynomials. If not, state why. If so, write in standard form and find the degree and leading coefficient.

1. $\frac{1}{x^2}+x+5$
2. $x^3+8x^4-15x+14x^2-20$
3. x^3+8
4. $5x^{-2}+9x^{-1}+16$
5. $x^2\sqrt{2}-x\sqrt{6}+10$
6. $\frac{x^4+8x^2+12}{3}$
7. $\frac{x^2-4}{x}$
8. $-6x^3+7x^5-10x^6+19x^2-3x+41$

Add or subtract the following polynomials.

9. $(x^3+8x^2-15x+11)+(3x^3-5x^2-4x+9)$
10. $(-2x^4+x^3+12x^2+6x-18)-(4x^4-7x^3+14x^2+18x-25)$
11. $(10x^3-x^2+6x+3)+(x^4-3x^3+8x^2-9x+16)$
12. $(7x^3-2x^2+4x-5)-(6x^4+10x^3+x^2+4x-1)$
13. $(15x^2+x-27)+(3x^3-12x+16)$
14. $(2x^5-3x^4+21x^2+11x-32)-(x^4-3x^3-9x^2+14x-15)$
15. $(8x^3-13x^2+24)-(x^3+4x^2-2x+17)+(5x^2+18x-19)$

7.3 Multiplication of Polynomials

Here you will learn how to multiply polynomials using the distributive property.

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

Guidance

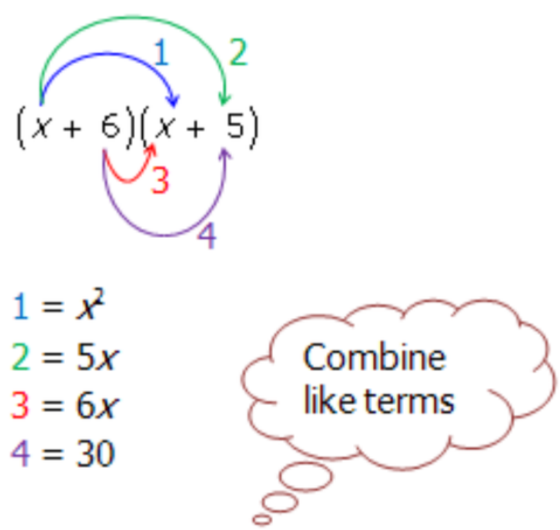
To multiply polynomials you will need to use the distributive property. Recall that the distributive property says that if you start with an expression like $3(5x+2)$, you can simplify it by multiplying both terms inside the parentheses by 3 to get a final answer of $15x+6$.

When multiplying polynomials, you will need to use the distributive property more than once for each problem.

Example A

Find the product: $(x+6)(x+5)$

Solution: To answer this question you will use the distributive property. The distributive property would tell you to multiply x in the first set of parentheses by everything inside the second set of parentheses , then multiply 6 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



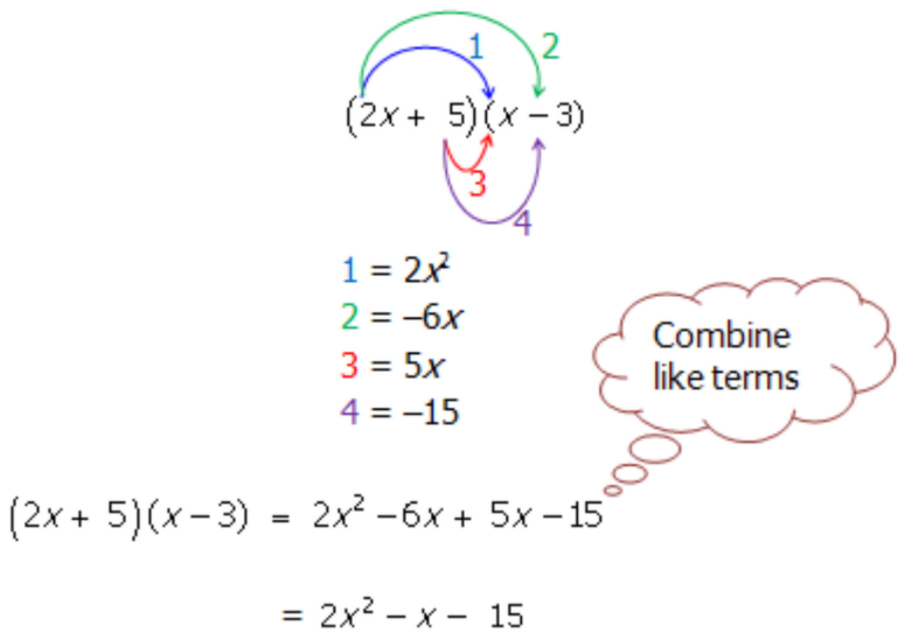
$$(x+6)(x+5) = x^2 + 5x + 6x + 30$$

$$= x^2 + 11x + 30$$

Example B

Find the product: $(2x+5)(x-3)$

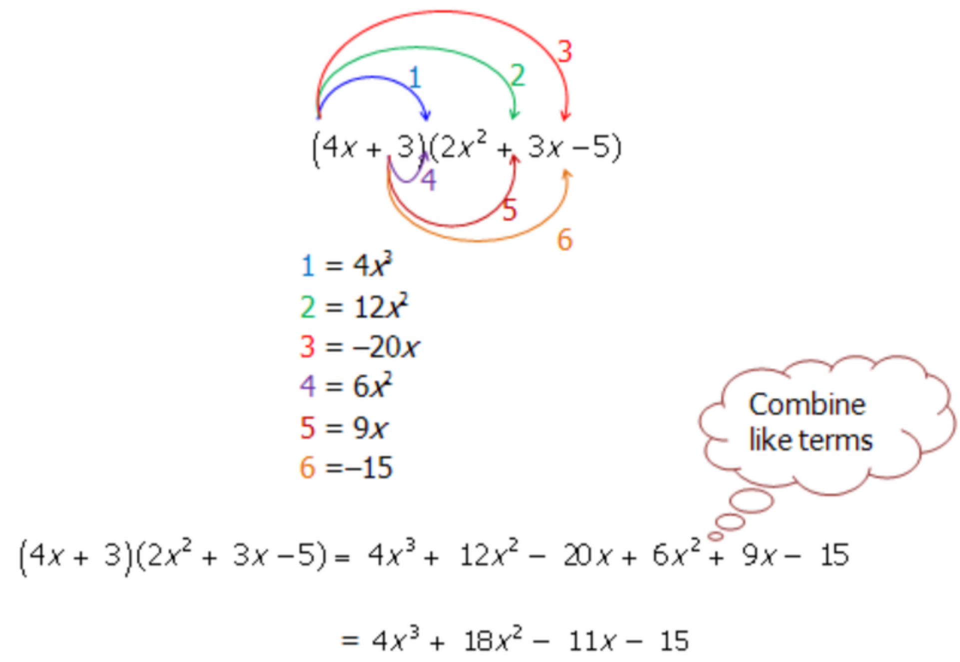
Solution: Again, use the distributive property. The distributive property tells you to multiply $2x$ in the first set of parentheses by everything inside the second set of parentheses , then multiply 5 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



Example C

Find the product: $(4x+3)(2x^2+3x-5)$

Solution: Even though at first this question may seem different, you can still use the distributive property to find the product. The distributive property tells you to multiply $4x$ in the first set of parentheses by everything inside the second set of parentheses, then multiply 3 in the first set of parentheses by everything in the second set of parentheses. Here is what that looks like:



Concept Problem Revisited

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

What is known?

The height of the picture frame is $4x+7$The equations:

- The width of the picture frame is $3x+5$

The formula:

$$\begin{aligned}
\text{Area} &= w \times h \\
\text{Area} &= (3x+5)(4x+7) \\
\text{Area} &= 12x^2 + 21x + 20x + 35 \\
\text{Area} &= 12x^2 + 41x + 35
\end{aligned}$$

Vocabulary

Distributive Property

The ***distributive property*** states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, in the expression: $\frac{2}{3}(x+5)$, the distributive property states that the product of a number $\left(\frac{2}{3}\right)$ and a sum $(x+5)$ is equal to the sum of the individual products of the number $\left(\frac{2}{3}\right)$ and the addends (x and 5).

Like Terms

Like terms refers to terms in which the degrees match and the variables match. For example $3x$ and $4x$ are like terms. Like terms are also known as **similar terms**.

Guided Practice

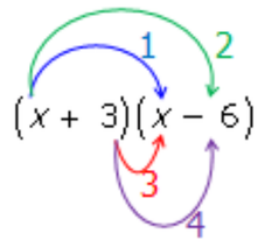
1. Find the product: $(x+3)(x-6)$

2. Find the product: $(2x+5)(3x^2-2x-7)$

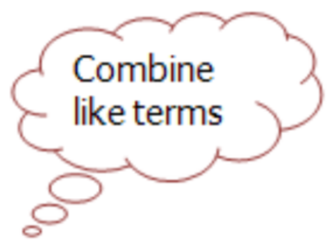
3. An average football field has the dimensions of 160 ft by 360 ft. If the expressions to find these dimensions were $(3x+7)$ and $(7x+3)$, what value of x would give the dimensions of the football field?

Answers:

1. $(x+3)(x-6)$

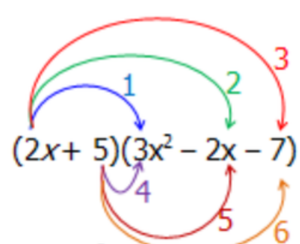


$1 = x^2$
$2 = -6x$
$3 = 3x$
$4 = -18$



$$\begin{aligned}(x+3)(x-6) &= x^2 - 6x + 3x - 18\\ &= x^2 - 3x - 18\end{aligned}$$

2. $(2x+5)(3x^2-2x-7)$



$1 = 6x^3$
$2 = -4x^2$
$3 = -14x$
$4 = 15x^2$
$5 = -10x$
$6 = -35$

$$\begin{aligned}(4x+3)(2x^2+3x-5) &= 6x^3 - 4x^2 - 14x + 15x^2 - 10x - 35\\ &= 6x^3 + 11x^2 - 24x - 35\end{aligned}$$

3. Area $= l \times w$

$$\begin{aligned}
\text{Area} &= 360 \times 160 \\
(7x+3) &= 360 \\
7x &= 360-3 \\
7x &= 357 \\
x &= 51 \\
\\
(3x+7) &= 160 \\
3x &= 160-7 \\
3x &= 153 \\
x &= 51
\end{aligned}$$

The value of x that satisfies these expressions is 51.

Practice

Use the distributive property to find the product of each of the following polynomials:

1. $(x+4)(x+6)$
2. $(x+3)(x+5)$
3. $(x+7)(x-8)$
4. $(x-9)(x-5)$
5. $(x-4)(x-7)$
6. $(x+3)(x^2+x+5)$
7. $(x+7)(x^2-3x+6)$
8. $(2x+5)(x^2-8x+3)$
9. $(2x-3)(3x^2+7x+6)$
10. $(5x-4)(4x^2-8x+5)$
11. $9a^2(6a^3+3a+7)$
12. $-4s^2(3s^3+7s^2+11)$
13. $(x+5)(5x^3+2x^2+3x+9)$
14. $(t-3)(6t^3+11t^2+22)$
15. $(2g-5)(3g^3+9g^2+7g+12)$

7.4 Special Products of Polynomials

Here you'll learn how to find two special polynomial products: 1) the square of a binomial and 2) two binomials where the sum and difference formula can be applied. You'll also learn how to apply special products of polynomials to solve real-world problems.

What if you wanted to multiply two binomials that were exactly the same, like $(x^2-2)(x^2-2)$? Similarly what if you wanted to multiply two binomials in which the sign between the two terms was the opposite in one from the other, like $(x^2-2)(x^2+2)$? What shortcuts could you use? After completing this Concept, you'll be able to find the square of a binomial as well as the product of binomials using the sum and difference formula.

Guidance

We saw that when we multiply two binomials we need to make sure to multiply each term in the first binomial with each term in the second binomial. Let's look at another example.

Multiply two linear binomials (binomials whose degree is 1):

$$(2x+3)(x+4)$$

When we multiply, we obtain a quadratic polynomial (one with degree 2) with four terms:

$$2x^2+8x+3x+12$$

The middle terms are like terms and we can combine them. We simplify and get $2x^2+11x+12$. This is a quadratic, or second-degree, **trinomial** (polynomial with three terms).

You can see that every time we multiply two linear binomials with one variable, we will obtain a quadratic polynomial. In this section we'll talk about some special products of binomials.

Find the Square of a Binomial

One special binomial product is the **square of a binomial**. Consider the product $(x+4)(x+4)$.

Since we are multiplying the same expression by itself, that means we are squaring the expression. $(x+4)(x+4)$ is the same as $(x+4)^2$.

When we multiply it out, we get $x^2+4x+4x+16$, which simplifies to $x^2+8x+16$.

Notice that the two middle terms—the ones we added together to get $8x$—were the same. Is this a coincidence? In order to find that out, let's square a general linear binomial.

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Sure enough, the middle terms are the same. How about if the expression we square is a difference instead of a sum?

$$\begin{aligned}(a-b)^2 = (a-b)(a-b) &= a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2\end{aligned}$$

It looks like the middle two terms are the same in general whenever we square a binomial. The general pattern is: to square a binomial, take the square of the first term, add or subtract twice the product of the terms, and add the square of the second term. You should remember these formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$

and

$$(a-b)^2 = a^2 - 2ab + b^2$$

Remember! Raising a polynomial to a power means that we multiply the polynomial by itself however many times the exponent indicates. For instance, $(a+b)^2 = (a+b)(a+b)$. **Don't make the common mistake of thinking that** $(a+b)^2 = a^2 + b^2$**!** To see why that's not true, try substituting numbers for a and b into the equation (for example, $a = 4$ and $b = 3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

We can apply the formulas for squaring binomials to any number of problems.

Example A

Square each binomial and simplify.

a) $(x+10)^2$

b) $(2x-3)^2$

c) $(x^2+4)^2$

Solution

Let's use the square of a binomial formula to multiply each expression.

a) $(x+10)^2$

If we let $a = x$ and $b = 10$, then our formula $(a+b)^2 = a^2 + 2ab + b^2$ becomes $(x+10)^2 = x^2 + 2(x)(10) + 10^2$, which simplifies to $x^2 + 20x + 100$.

b) $(2x-3)^2$

If we let $a = 2x$ and $b = 3$, then our formula $(a-b)^2 = a^2 - 2ab + b^2$ becomes $(2x-3)^2 = (2x^2) - 2(2x)(3) + (3)^2$, which simplifies to $4x^2 - 12x + 9$.

c) $(x^2+4)^2$

If we let $a = x^2$ and $b = 4$, then

$$\begin{aligned}(x^2+4)^2 &= (x^2)^2 + 2(x^2)(4) + (4)^2 \\ &= x^4 + 8x^2 + 16\end{aligned}$$

Find the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$\begin{aligned}(x+4)(x-4) &= x^2-4x+4x-16\\ &= x^2-16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they *cancel out* when we collect like terms. This is not a coincidence. This always happens when we multiply a sum and difference of the same terms. In general,

$$\begin{aligned}(a+b)(a-b) &= a^2-ab+ab-b^2\\ &= a^2-b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula: $(a+b)(a-b)=a^2-b^2$

Let's apply this formula to a few examples.

Example B

Multiply the following binomials and simplify.

a) $(x+3)(x-3)$

b) $(5x+9)(5x-9)$

c) $(2x^3+7)(2x^3-7)$

Solution

a) Let $a=x$ and $b=3$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\ (x+3)(x-3) &= x^2-3^2\\ &= x^2-9\end{aligned}$$

b) Let $a=5x$ and $b=9$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\ (5x+9)(5x-9) &= (5x)^2-9^2\\ &= 25x^2-81\end{aligned}$$

c) Let $a=2x^3$ and $b=7$, then:

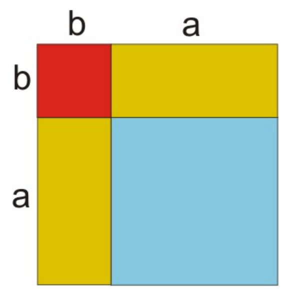
$$\begin{aligned}(2x^3+7)(2x^3-7) &= (2x^3)^2-(7)^2\\ &= 4x^6-49\end{aligned}$$

Solve Real-World Problems Using Special Products of Polynomials

Now let's see how special products of polynomials apply to geometry problems and to mental arithmetic.

Example C

Find the area of the following square:



Solution

The length of each side is $(a+b)$, so the area is $(a+b)(a+b)$.

Notice that this gives a visual explanation of the square of a binomial. The blue square has area a^2, the red square has area b^2, and each rectangle has area ab, so added all together, the area $(a+b)(a+b)$ is equal to $a^2+2ab+b^2$.

The next example shows how you can use the special products to do fast mental calculations.

Example D

Use the difference of squares and the binomial square formulas to find the products of the following numbers without using a calculator.

a) 43×57

b) 45^2

c) 481×319

Solution

The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

a) Rewrite 43 as $(50-7)$ and 57 as $(50+7)$.

Then $43 \times 57 = (50-7)(50+7) = (50)^2-(7)^2 = 2500-49 = 2451$

b) $45^2 = (40+5)^2 = (40)^2+2(40)(5)+(5)^2 = 1600+400+25 = 2025$

c) Rewrite 481 as $(400+81)$ and 319 as $(400-81)$.

Then $481 \times 319 = (400+81)(400-81) = (400)^2-(81)^2$

$(400)^2$ is easy - it equals 160000.

$(81)^2$ is not easy to do mentally, so let's rewrite 81 as $80+1$.

$(81)^2 = (80+1)^2 = (80)^2+2(80)(1)+(1)^2 = 6400+160+1 = 6561$

Then $481 \times 319 = (400)^2-(81)^2 = 160000-6561 = 153439$

Vocabulary

- **Square of a binomial:** $(a+b)^2 = a^2+2ab+b^2$, *and* $(a-b)^2 = a^2-2ab+b^2$
- **Sum and difference formula:** $(a+b)(a-b) = a^2-b^2$

Guided Practice

1. *Square the binomial and simplify:* $(5x-2y)^2$.

2. *Multiply* $(4x+5y)(4x-5y)$ *and simplify.*

3. *Use the difference of squares and the binomial square formulas to find the product of* 112×88 *without using a calculator.*

Solutions:

1.) $(5x-2y)^2$

If we let $a = 5x$ and $b = 2y$, then

$$\begin{aligned}(5x-2y)^2 &= (5x)^2 - 2(5x)(2y) + (2y)^2 \\ &= 25x^2 - 20xy + 4y^2\end{aligned}$$

2.) Let $a = 4x$ and $b = 5y$, then:

$$\begin{aligned}(4x+5y)(4x-5y) &= (4x)^2 - (5y)^2 \\ &= 16x^2 - 25y^2\end{aligned}$$

3.) The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

Rewrite 112 as $(100+12)$ and 88 as $(100-12)$.

Then

$$\begin{aligned}112 \times 88 &= (100+12)(100-12) \\ &= (100)^2 - (12)^2 \\ &= 10000 - 144 \\ &= 9856\end{aligned}$$

Practice

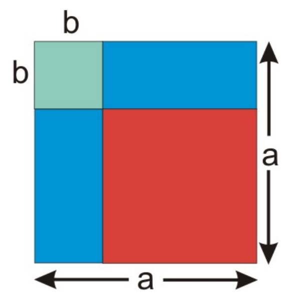
Use the special product rule for squaring binomials to multiply these expressions.

1. $(x+9)^2$
2. $(3x-7)^2$
3. $(5x-y)^2$
4. $(2x^3-3)^2$
5. $(4x^2+y^2)^2$
6. $(8x-3)^2$
7. $(2x+5)(5+2x)$
8. $(xy-y)^2$

Use the special product of a sum and difference to multiply these expressions.

9. $(2x-1)(2x+1)$
10. $(x-12)(x+12)$
11. $(5a-2b)(5a+2b)$
12. $(ab-1)(ab+1)$
13. $(z^2+y)(z^2-y)$
14. $(2q^3+r^2)(2q^3-r^2)$
15. $(7s-t)(t+7s)$
16. $(x^2y+xy^2)(x^2y-xy^2)$

Find the area of the lower right square in the following figure.

17. 

Multiply the following numbers using special products.

18. 45×55
19. 56^2
20. 1002×998
21. 36×44
22. 10.5×9.5
23. 100.2×9.8
24. -95×-105
25. 2×-2

7.5 Monomial Factors of Polynomials

Here you will learn to find a common factor in a polynomial and factor it out of the polynomial.

Can you write the following polynomial as a product of a monomial and a polynomial?

$12x^4+6x^3+3x^2$

Guidance

In the past you have studied common factors of two numbers. Consider the numbers 25 and 35. A common factor of 25 and 35 is 5 because 5 goes into both 25 and 35 evenly.

This idea can be extended to polynomials. A common factor of a polynomial is a number and/or variable that are a factor in all terms of the polynomial. The Greatest Common Factor (or GCF) is the largest monomial that is a factor of each of the terms of the polynomial.

To factor a polynomial means to write the polynomial as a product of other polynomials. One way to factor a polynomial is:

1. Look for the greatest common factor.
2. Write the polynomial as a product of the **greatest common factor** and **the polynomial that results when you divide all the terms of the original polynomial by the greatest common factor.**

One way to think about this type of factoring is that you are essentially doing the distributive property in reverse.

Example A

Factor the following binomial: $5a+15$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5 and 15 can both be divided by 5. The GCF for this binomial is 5.

Step 2: Divide the GCF out of each term of the binomial:

$5a+15=5(a+3)$

Example B

Factor the following polynomial: $4x^2+8x-2$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 4, 8 and 2 can all be divided by 2. The GCF for this polynomial is 2.

Step 2: Divide the GCF out of each term of the polynomial:

$4x^2+8x-2=2(2x^2+4x-1)$

Example C

Factor the following polynomial: $3x^5 - 9x^3 - 6x^2$

Solution: *Step 1:* Identify the GCF. Looking at each of the terms, you can see that 3, 9 and 6 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$3x^5 - 9x^3 - 6x^2 = 3x^2(x^3 - 3x - 2)$

Concept Problem Revisited

To write as a product you want to try to factor the polynomial: $12x^4 + 6x^3 + 3x^2$.

Step 1: Identify the GCF of the polynomial. Looking at each of the numbers, you can see that 12, 6, and 3 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$12x^4 + 6x^3 + 3x^2 = 3x^2(4x^2 + 2x + 1)$$

Vocabulary

Common Factor
: ***Common factors*** are numbers (numerical coefficients) or letters (literal coefficients) that are a factor in all parts of the polynomials.

Greatest Common Factor
: The ***Greatest Common Factor*** (or ***GCF***) is the largest monomial that is a factor of (or divides into evenly) each of the terms of the polynomial.

Guided Practice

1. Find the common factors of the following: $a^2(b+7) - 6(b+7)$
2. Factor the following polynomial: $5k^6 + 15k^4 + 10k^3 + 25k^2$
3. Factor the following polynomial: $27x^3y + 18x^2y^2 + 9xy^3$

Answers:

1. *Step 1:* Identify the GCF

This problem is a little different in that if you look at the expression you notice that $(b+7)$ is common in both terms. Therefore $(b+7)$ is the common factor. The GCF for this expression is $(b+7)$.

Step 2: Divide the GCF out of each term of the expression:

$$a^2(b+7) - 6(b+7) = (a^2 - 6)(b+7)$$

2. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5, 15, 10, and 25 can all be divided by 5. Also notice that each of the terms has an k^2 in common. The GCF for this polynomial is $5k^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$5k^6 + 15k^4 + 10k^3 + 25k^2 = 5k^2(k^4 + 3k^2 + 2k + 5)$$

3. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 27, 18 and 9 can all be divided by 9. Also notice that each of the terms has an xy in common. The GCF for this polynomial is $9xy$.

Step 2: Divide the GCF out of each term of the polynomial:
$27x^3y + 18x^2y^2 + 9xy^3 = 9xy(3x^2 + 2xy + y^2)$

Practice

Factor the following polynomials by looking for a common factor:

1. $7x^2 + 14$
2. $9c^2 + 3$
3. $8a^2 + 4a$
4. $16x^2 + 24y^2$
5. $2x^2 - 12x + 8$
6. $32w^2x + 16xy + 8x^2$
7. $12abc + 6bcd + 24acd$
8. $15x^2y - 10x^2y^2 + 25x^2y$
9. $12a^2b - 18ab^2 - 24a^2b^2$
10. $4s^3t^2 - 16s^2t^3 + 12st^2 - 24st^3$

Find the common factors of the following expressions and then factor:

1. $2x(x-5) + 7(x-5)$
2. $4x(x-3) + 5(x-3)$
3. $3x^2(e+4) - 5(e+4)$
4. $8x^2(c-3) - 7(c-3)$
5. $ax(x-b) + c(x-b)$

7.6 Factoring When the Leading Coefficient Equals 1

Here you'll learn how to factor a quadratic equation in standard form, when $a = 1$.

The area of a rectangle is $x^2 - 3x - 28$. What are the length and width of the rectangle?

Guidance

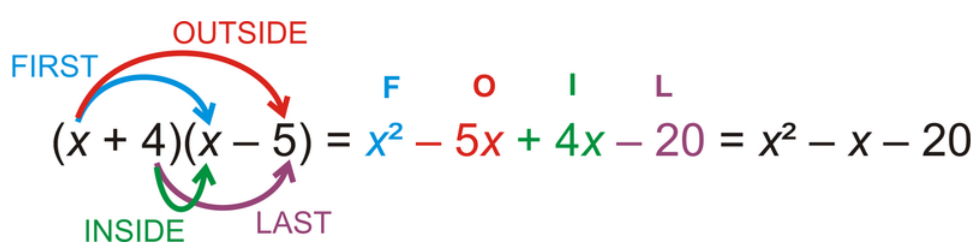
A **quadratic equation** has the form $ax^2 + bx + c$, where $a \neq 0$ (If $a = 0$, then the equation would be linear). For all quadratic equations, the 2 is the largest and only exponent. A quadratic equation can also be called a **trinomial** when all three terms are present.

There are four ways to solve a quadratic equation. The easiest is **factoring**. In this concept, we are going to focus on factoring when $a = 1$ or when there is no number in front of x^2. First, let's start with a review of multiplying two factors together.

Example A

Multiply $(x+4)(x-5)$.

Solution: Even though this is not a quadratic, the product of the two **factors** will be. Remember from previous math classes that a factor is a number that goes evenly into a larger number. For example, 4 and 5 are factors of 20. So, to determine the larger number that $(x+4)$ and $(x-5)$ go into, we need to multiply them together. One method for multiplying two polynomial factors together is called FOIL. To do this, you need to multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and the LAST terms together and then combine like terms.



Therefore $(x+4)(x-5) = x^2 - x - 20$. We can also say that $(x+4)$ and $(x-5)$ are factors of $x^2 - x - 20$.

More Guidance

Now, we will "undo" the multiplication of two factors by factoring. In this concept, we will only address quadratic equations in the form $x^2 + bx + c$, or when $a = 1$.

Investigation: Factoring

1. From the previous example, we know that $(x+m)(x+n) = x^2 + bx + c$.

FOIL $(x+m)(x+n)$.

$$(x+m)(x+n) \Rightarrow x^2 + \underbrace{nx+mx}_{bx} + \underbrace{mn}_{c}$$

2. This shows us that the **constant** term, or c, is equal to the product of the constant numbers inside each factor. It also shows us that the **coefficient** in front of x, or b, is equal to the sum of these numbers.

3. Group together the first two terms and the last two terms. Find the Greatest Common Factor, or GCF, for each pair.

$$(x^2+nx)+(mx+mn)$$
$$x(x+n)+m(x+n)$$

4. Notice that what is inside both sets of parenthesis in Step 3 is the same. This number, $(x+n)$, is the GCF of $x(x+n)$ and $m(x+n)$. You can pull it out in front of the two terms and leave the $x+m$.

$$x(x+n)+m(x+n)$$
$$(x+n)(x+m)$$

We have now shown how to go from FOIL-ing to factoring and back. Let's apply this idea to an example.

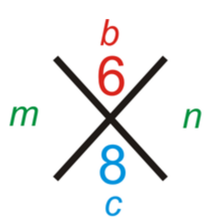
Example B

Factor x^2+6x+8.

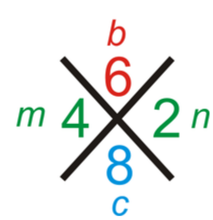
Solution: Let's use the investigation to help us.

$$x^2+6x+8=(x+m)(x+n)$$

So, from Step 2, b will be equal to the sum of m and n and c will be equal to their product. Applying this to our problem, $6=m+n$ and $8=mn$. To organize this, use an "X". Place the sum in the top and the product in the bottom.



Factors of 8	Sum
4 · 2	6
-4 · -2	-6
1 · 8	9
-1 · -8	-9



The green pair above is the only one that also adds up to 6. Now, move on to Step 3 from our investigation. We need to rewrite the $x-$term, or b, as a sum of m and n.

$$x^2+6x+8$$
$$x^2+4x+2x+8$$
$$(x^2+4x)+(2x+8)$$
$$x(x+4)+2(x+4)$$

Moving on to Step 4, we notice that the $(x+4)$ term is the same. Pull this out and we are done.

$$x(x+4)+2(x+4)$$
$$(x+4)(x+2)$$

Therefore, the factors of x^2+6x+8 are $(x+4)(x+2)$. You can FOIL this to check your answer.

Example C

Factor $x^2+12x-28$.

Solution: We can approach this problem in exactly the same way we did Example B. This time, we will not use the "X." What are the factors of -28 that also add up to 12? Let's list them out to see:

$$-4\cdot 7, 4\cdot -7, 2\cdot -14, -2\cdot 14, 1\cdot -28, -1\cdot 28$$

The **red** pair above is the one that works. Notice that we only listed the factors of *negative* 28.

$$\begin{aligned}
&x^2+12x-28\\
&x^2-2x+14x-28\\
&(x^2-2x)+(14x-28)\\
&x(x-2)+14(x-2)\\
&(x-2)(x+14)
\end{aligned}$$

By now, you might have a couple questions:

1. Does it matter which $x-$term you put first? NO, order does not matter. In the previous example, we could have put $14x$ followed by $-2x$. We would still end up with the same answer.
2. Can I skip the "expanded" part (Steps 3 and 4 in the investigation)? YES and NO. Yes, if $a=1$ No, if $a\neq 1$ (the next concept). If $a=1$, then $x^2+bx+c=(x+m)(x+n)$ such that $m+n=b$ and $mn=c$. Consider this a shortcut.

Example D

Factor x^2-4x.

Solution: This is an example of a quadratic that is not a trinomial because it only has two terms, also called a **binomial**. There is no c, or constant term. To factor this, we need to look for the GCF. In this case, the largest number that can be taken out of both terms is an x.

$$x^2-4x=x(x-4)$$

Therefore, the factors are x and $x-4$.

Intro Problem Revisit Recall that the area of a rectangle is $A = lw$, where l is the length and w is the width. To find the length and width, we can therefore factor the area $x^2 - 3x - 28$.

What are the factors of –28 that add up to –3? Testing the various possibilities, we find that $-7 \cdot 4 = -28$ and $-7 + 4 = -3$.

Therefore, $x^2 - 3x - 28$ factors to $(x-7)(x+4)$, and one of these factors is the rectangle's length while the other is its width.

Guided Practice

1. Multiply $(x-3)(x+8)$.

Factor the following quadratics, if possible.

2. $x^2 - 9x + 20$

3. $x^2 + 7x - 30$

4. $x^2 + x + 6$

5. $x^2 + 10x$

Answers

1. FOIL-ing our factors together, we get:

$$(x-3)(x+8) = x^2 + 8x - 3x - 24 = x^2 + 5x - 24$$

2. Using the "X," we have:

From the shortcut above, $-4 + -5 = -9$ and $-4 \cdot -5 = 20$.

$$x^2 - 9x + 20 = (x-4)(x-5)$$

3. Let's list out all the factors of -30 and their sums. The sums are in red.

$$-10 \cdot 3\ (-7), -3 \cdot 10\ (7), -2 \cdot 15\ (13), -15 \cdot 2\ (-13), -1 \cdot 30\ (29), -30 \cdot 1\ (-29)$$

From this, the factors of -30 that add up to 7 are -3 and 10. $x^2 + 7x - 30 = (x-3)(x+10)$

4. There are no factors of 6 that add up to 1. If we had -6, then the trinomial would be factorable. But, as is, this is not a factorable trinomial.

5. The only thing we can do here is to take out the GCF. $x^2 + 10x = x(x+10)$

Vocabulary

Quadratic Equation
An equation where the largest exponent is a 2 and has the form ax^2+bx+c, $a \neq 0$.

Trinomial
A quadratic equation with three terms.

Binomial
A quadratic equation with two terms.

Factoring
A way to break down a quadratic equation into smaller factors.

Factor
A number that goes evenly into a larger number.

FOIL
A method used to multiply together two factors. You multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and LAST terms and then combine any like terms.

Coefficient
The number in front of a variable.

Constant
A number that is added or subtracted within an equation.

Practice

Multiply the following factors together.

1. $(x+2)(x-8)$
2. $(x-9)(x-1)$
3. $(x+7)(x+3)$

Factor the following quadratic equations. If it cannot be factored, write *not factorable*. You can use either method presented in the examples.

4. x^2-x-2
5. $x^2+2x-24$
6. x^2-6x
7. x^2+6x+9
8. $x^2+8x-10$
9. $x^2-11x+30$
10. $x^2+13x-30$
11. $x^2+11x+28$
12. $x^2-8x+12$
13. $x^2-7x-44$
14. $x^2-8x-20$

15. x^2+4x+3
16. $x^2-5x+36$
17. $x^2-5x-36$
18. x^2+x

Challenge Fill in the X's below with the correct numbers.

19. 

20. 

7.7 Factoring When the Leading Coefficient Doesn't Equal 1

Here you'll learn how to factor a quadratic equation in standard form, by expanding the x-term.

The area of a square is $9x^2+24x+16$. What are the dimensions of the square?

Guidance

When we add a number in front of the x^2 term, it makes factoring a little trickier. We still follow the investigation from the previous section, but ***cannot*** use the shortcut. First, let's try FOIL-ing when the coefficients in front of the $x-$terms are not 1.

Example A

Multiply $(3x-5)(2x+1)$

Solution: We can still use FOIL.

FIRST $3x \cdot 2x = 6x^2$

OUTSIDE $3x \cdot 1 = 3x$

INSIDE $-5 \cdot 2x = -10x$

LAST $-5 \cdot 1 = -5$

Combining all the terms together, we get: $6x^2+3x-10x-5 = 6x^2-7x-5$.

Now, let's work backwards and factor a trinomial to get two factors. Remember, you can always check your work by multiplying the final factors together.

Example B

Factor $6x^2-x-2$.

Solution: This is a factorable trinomial. When there is a coefficient, or number in front of x^2, you must follow all the steps from the investigation in the previous concept; no shortcuts. Also, m and n no longer have a product of c and a sum of b. This would not take the coefficient of x^2 into account. What we need to do is multiply together a and c (from ax^2+bx+c) and then find the two numbers whose product is ac and sum is b. Let's use the X to help us organize this.

Now, we can see, we need the two factors of -12 that also add up to -1.

TABLE 7.1:

Factors	Sum
$-1, 12$	11
1, -12	-11
2, -6	-4
-2, 6	4
$3, -4$	-1
-3, 4	1

The factors that work are 3 and -4. Now, take these factors and rewrite the $x-$term expanded using 3 and -4 (Step 3 from the investigation in the previous concept).

$$6x^2 - x - 2$$

$$6x^2 - 4x + 3x - 2$$

Next, group the first two terms together and the last two terms together and pull out any common factors.

$$(6x^2 - 4x) + (3x - 2)$$
$$2x(3x - 2) + 1(3x - 2)$$

Just like in the investigation, what is in the parenthesis is *the same*. We now have two terms that both have $(3x - 2)$ as factor. Pull this factor out.

$$2x(3x - 2) + 1(3x - 2)$$
$$(3x - 2)(2x + 1)$$

The factors of $6x^2 - x - 2$ are $(3x - 2)(2x + 1)$. You can FOIL these to check your answer.

Example C

Factor $4x^2 + 8x - 5$.

Solution: Let's make the steps from Example B a little more concise.

1. Find *ac* and the factors of this number that add up to *b*.

$4 \cdot -5 = -20$ The factors of -20 that add up to 8 are 10 and -2.

2. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 1.

$$4x^2 + 8x - 5$$

$$4x^2 + 10x - 2x - 5$$

3. Group the first two and second two terms together, find the GCF and factor again.

$$\begin{aligned}&(4x^2+10x)+(-2x-5)\\&2x(2x+5)-1(2x+5)\\&(2x+5)(2x-1)\end{aligned}$$

Alternate Method: What happens if we list $-2x$ before $10x$ in Step 2?

$$\begin{aligned}&4x^2-2x+10x-5\\&(4x^2-2x)(10x-5)\\&2x(2x-1)+5(2x-1)\\&(2x-1)(2x+5)\end{aligned}$$

This tells us it does not matter which $x-$term we list first in Step 2 above.

Example D

Factor $12x^2-22x-20$.

Solution: Let's use the steps from Example C, but we are going to add an additional step at the beginning.

1. Look for any common factors. Pull out the GCF of all three terms, if there is one.

$$12x^2-22x-20=2(6x^2-11x-10)$$

This will make it much easier for you to factor what is inside the parenthesis.

2. Using what is inside the parenthesis, find ac and determine the factors that add up to b.

$$6\cdot -10=-60 \rightarrow -15\cdot 4=-60,\ -15+4=-11$$

The factors of -60 that add up to -11 are -15 and 4.

3. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 2.

$$\begin{aligned}&2(6x^2-11x-10)\\&2(6x^2-15x+4x-10)\end{aligned}$$

4. Group the first two and second two terms together, find the GCF and factor again.

$$\begin{aligned}&2(6x^2-15x+4x-10)\\&2\left[(6x^2-15x)+(4x-10)\right]\\&2\left[3x(2x-5)+2(2x-5)\right]\\&2(2x-5)(3x+2)\end{aligned}$$

Intro Problem Revisit The dimensions of a square are its length and its width, so we need to factor the area $9x^2+24x+16$.

We need to multiply together a and c (from ax^2+bx+c) and then find the two numbers whose product is ac and whose sum is b.

Now we can see that we need the two factors of 144 that also add up to 24. Testing the possibilities, we find that $12 \cdot 12 = 144$ and $12+12 = 24$.

Now, take these factors and rewrite the $x-$term expanded using 12 and 12.

$$9x^2 + 24x + 16$$

$$9x^2 + 12x + 12x + 16$$

Next, group the first two terms together and the last two terms together and pull out any common factors.

$$\begin{aligned}&(9x^2+12x)+(12x+16)\\&3x(3x+4)+4(3x+4)\end{aligned}$$

We now have two terms that both have $(3x+4)$ as factor. Pull this factor out.

The factors of $9x^2+24x+16$ are $(3x+4)(3x+4)$, which are also the dimensions of the square.

Guided Practice

1. Multiply $(4x-3)(3x+5)$.

Factor the following quadratics, if possible.

2. $15x^2-4x-3$

3. $3x^2+6x-12$

4. $24x^2-30x-9$

5. $4x^2+4x-48$

Answers

1. FOIL: $(4x-3)(3x+5) = 12x^2+20x-9x-15 = 12x^2+11x-15$

2. Use the steps from the examples above. There is no GCF, so we can find the factors of ac that add up to b.

$15 \cdot -3 = -45$ The factors of -45 that add up to -4 are -9 and 5.

$$\begin{aligned}&15x^2-4x-3\\&(15x^2-9x)+(5x-3)\\&3x(5x-3)+1(5x-3)\\&(5x-3)(3x+1)\end{aligned}$$

3. $3x^2+6x-12$ has a GCF of 3. Pulling this out, we have $3(x^2+2x-6)$. There is no number in front of x^2, so we see if there are any factors of -6 that add up to 2. There are not, so this trinomial is not factorable.

4. $24x^2-30x-9$ also has a GCF of 3. Pulling this out, we have $3(8x^2-10x-3)$. $ac=-24$. The factors of -24 than add up to -10 are -12 and 2.

$$3(8x^2-10x-3)$$
$$3\left[(8x^2-12x)+(2x-3)\right]$$
$$3\left[4x(2x-3)+1(2x-3)\right]$$
$$3(2x-3)(4x+1)$$

5. $4x^2+4x-48$ has a GCF of 4. Pulling this out, we have $4(x^2+x-12)$. This trinomial does not have a number in front of x^2, so we can use the shortcut from the previous concept. What are the factors of -12 that add up to 1?

$$4(x^2+x-12)$$
$$4(x+4)(x-3)$$

Practice

Multiply the following expressions.

1. $(2x-1)(x+5)$
2. $(3x+2)(2x-3)$
3. $(4x+1)(4x-1)$

Factor the following quadratic equations, if possible. If they cannot be factored, write *not factorable*. Don't forget to look for any GCFs first.

4. $5x^2+18x+9$
5. $6x^2-21x$
6. $10x^2-x-3$
7. $3x^2+2x-8$
8. $4x^2+8x+3$
9. $12x^2-12x-18$
10. $16x^2-6x-1$
11. $5x^2-35x+60$
12. $2x^2+7x+3$
13. $3x^2+3x+27$
14. $8x^2-14x-4$
15. $10x^2+27x-9$
16. $4x^2+12x+9$
17. $15x^2+35x$
18. $6x^2-19x+15$
19. Factor x^2-25. What is b?
20. Factor $9x^2-16$. What is b? What types of numbers are a and c?

7.8 Factoring Special Quadratics

Here you'll learn to factor perfect square trinomials and the difference of squares.

The total time, in hours, it takes a rower to paddle upstream, turn around and come back to her starting point is $18x^2 = 32$. How long does it take her to make the round trip?

Guidance

There are a couple of special quadratics that, when factored, have a pattern.

Investigation: Multiplying the Square of a Binomal

1. Rewrite $(a+b)^2$ as the product of two factors. Expand $(a+b)^2$. $(a+b)^2 = (a+b)(a+b)$
2. FOIL your answer from Step 1. This is a **perfect square trinomial.** $a^2+2ab+b^2$
3. $(a-b)^2$ also produces a perfect square trinomial. $(a-b)^2 = a^2-2ab+b^2$
4. Apply the formula above to factoring $9x^2-12x+4$. First, find a and b.

$$\begin{aligned} a^2 &= 9x^2, \ b^2 = 4 \\ a &= 3x, \quad b = 2 \end{aligned}$$

5. Now, plug a and b into the appropriate formula.

$$\begin{aligned} (3x-2)^2 &= (3x)^2 - 2(3x)(2) + 2^2 \\ &= 9x^2 - 12x + 4 \end{aligned}$$

Investigation: Multiplying (a b)(a - b)

1. FOIL $(a-b)(a+b)$.

$$\begin{aligned} (a-b)(a+b) &= a^2 + ab - ab - b^2 \\ &= a^2 - b^2 \end{aligned}$$

2. This is a **difference of squares.** The difference of squares will always factor to be $(a+b)(a-b)$.
3. Apply the formula above to factoring $25x^2-16$. First, find a and b.

$$\begin{aligned} a^2 &= 25x^2, \ b^2 = 16 \\ a &= 5x, \quad b = 4 \end{aligned}$$

4. Now, plug a and b into the appropriate formula. $(5x-4)(5x+4)=(5x)^2-4^2$

**It is important to note that if you forget these formulas or do not want to use them, you can still factor all of these quadratics the same way you did in the previous two concepts.

Example A

Factor x^2-81.

Solution: Using the formula from the investigation above, we need to first find the values of a and b.

$$\begin{aligned} x^2-81 &= a^2-b^2 \\ a^2=x^2&,\ b^2=81 \\ a=x&, \quad b=9 \end{aligned}$$

Now, plugging x and 9 into the formula, we have $x^2-81=(x-9)(x+9)$. To solve for a and b, we found the **square root** of each number. Recall that the square root is a number that, when multiplied by itself, produces another number. This other number is called a **perfect square**.

Alternate Method

Rewrite x^2-81 so that the middle term is present. $x^2+0x-81$

Using the method from the previous two concepts, what are the two factors of -81 that add up to 0? 9 and -9

Therefore, the factors are $(x-9)(x+9)$.

Example B

Factor $36x^2+120x+100$.

Solution: First, check for a GCF.

$$4(9x^2+30x+25)$$

Now, double-check that the quadratic equation above fits into the perfect square trinomial formula.

$$\begin{array}{ccc} a^2=9x^2 & b^2=25 & \\ \sqrt{a^2}=\sqrt{9x^2} & \sqrt{b^2}=\sqrt{25} & 2ab=30x \\ a=3x & b=5 & 2(3x)(5)=30x \end{array}$$

Using a and b above, the equation factors to be $4(3x+5)^2$. If you did not factor out the 4 in the beginning, the formula will still work. a would equal $6x$ and b would equal 10, so the factors would be $(6x+10)^2$. If you expand and find the GCF, you would have $(6x+10)^2=(6x+10)(6x+10)=2(3x+5)2(3x+5)=4(3x+5)^2$.

Alternate Method

First, find the GCF. $4(9x^2+30x+25)$

Then, find ac and expand b accordingly. $9\cdot 25=225$, the factors of 225 that add up to 30 are 15 and 15.

$$\begin{aligned}
&4(9x^2+30x+25)\\
&4(9x^2+15x+15x+25)\\
&4\left[(9x^2+15x)+(15x+25)\right]\\
&4\left[3x(3x+5)+5(3x+5)\right]\\
&4(3x+5)(3x+5)\ or\ 4(3x^2+5)
\end{aligned}$$

Again, notice that if you do not use the formula discovered in this concept, you can still factor and get the correct answer.

Example C

Factor $48x^2-147$.

Solution: At first glance, this does not look like a difference of squares. 48 nor 147 are square numbers. But, if we take a 3 out of both, we have $3(16x^2-49)$. 16 and 49 are both square numbers, so now we can use the formula.

$$\begin{aligned}
16x^2&=a^2 & 49&=b^2\\
4x&=a & 7&=b
\end{aligned}$$

The factors are $3(4x-7)(4x+7)$.

Intro Problem Revisit $18x^2=32$ can be rewritten as $18x^2-32=0$, so factor $18x^2-32$.

First, we must take greatest common factor of 2 out of both. We then have $2(9x^2-16)$. 9 and 16 are both square numbers, so now we can use the formula.

$$\begin{aligned}
9x^2&=a^2 & 16&=b^2\\
3x&=a & 4&=b
\end{aligned}$$

The factors are $2(3x-4)(3x+4)$.

Finally, to find the time, set these factors equal to zero and solve $2(3x-4)(3x+4)=0$.

Because x represents the time, it must be positive. Only $(3x-4)=0$ results in a positive value of x.

$x=\frac{4}{3}=1.3333$ Therefore the round trip takes 1.3333 hours.

You will do more problems like this one in the next lesson.

Guided Practice

Factor the following quadratic equations.

1. x^2-4

2. $2x^2-20x+50$

3. $81x^2+144+64$

Answers

1. $a = x$ and $b = 2$. Therefore, $x^2 - 4 = (x-2)(x+2)$.

2. Factor out the GCF, 2. $2(x^2 - 10x + 25)$. This is now a perfect square trinomial with $a = x$ and $b = 5$.

$$2(x^2 - 10x + 25) = 2(x-5)^2.$$

3. This is a perfect square trinomial and no common factors. Solve for a and b.

$$\begin{aligned} 81x^2 &= a^2 & 64 &= b^2 \\ 9x &= a & 8 &= b \end{aligned}$$

The factors are $(9x+8)^2$.

Vocabulary

Perfect Square Trinomial
A quadratic equation in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$.

Difference of Squares
A quadratic equation in the form $a^2 - b^2$.

Square Root
A number, that when multiplied by itself produces another number. 3 is the square root of 9.

Perfect Square
A number that has a square root that is an integer. 25 is a perfect square.

Practice

1. List the perfect squares that are less than 200.
2. Why do you think there is no *sum of squares* formula?

Factor the following quadratics, if possible.

3. $x^2 - 1$
4. $x^2 + 4x + 4$
5. $16x^2 - 24x + 9$
6. $-3x^2 + 36x - 108$
7. $144x^2 - 49$
8. $196x^2 + 140x + 25$
9. $100x^2 + 1$
10. $162x^2 + 72x + 8$
11. $225 - x^2$
12. $121 - 132x + 36x^2$
13. $5x^2 + 100x - 500$

14. $256x^2 - 676$
15. **Error Analysis** Spencer is given the following problem: Multiply $(2x-5)^2$. Here is his work:

$$(2x-5)^2 = (2x)^2 - 5^2 = 4x^2 - 25$$

His teacher tells him the answer is $4x^2 - 20x + 25$. What did Spencer do wrong? Describe his error and correct the problem.

7.9 Zero Product Principle

Here you'll learn how to apply the zero-product property and how to factor polynomials to solve for their unknown variables.

What if you had a polynomial equation like $3x^2+4x-4=0$? How could you factor the polynomial to solve the equation? After completing this Concept, you'll be able to solve polynomial equations by factoring and by using the zero-product property.

Guidance

The most useful thing about factoring is that we can use it to help solve polynomial equations.

Example A

Consider an equation like $2x^2+5x-42=0$. How do you solve for x?

Solution:

There's no good way to isolate x in this equation, so we can't solve it using any of the techniques we've already learned. But the left-hand side of the equation can be factored, making the equation $(x+6)(2x-7)=0$.

How is this helpful? The answer lies in a useful property of multiplication: if two numbers multiply to zero, then at least one of those numbers must be zero. This is called the **Zero-Product Property.**

What does this mean for our polynomial equation? Since the product equals 0, then at least one of the factors on the left-hand side must equal zero. So we can find the two solutions by setting each factor equal to zero and solving each equation separately.

Setting the factors equal to zero gives us:

$$(x+6)=0 \qquad \text{OR} \qquad (2x-7)=0$$

Solving both of those equations gives us:

$$\begin{array}{lcl} x+6=0 & & 2x-7=0 \\ \underline{\underline{x=-6}} & \text{OR} & 2x=7 \\ & & \underline{\underline{x=\frac{7}{2}}} \end{array}$$

Notice that the solution is $x=-6$ **OR** $x=\frac{7}{2}$. The **OR** means that either of these values of x would make the product of the two factors equal to zero. Let's plug the solutions back into the equation and check that this is correct.

$$\begin{aligned}
&Check: x = -6; \\
&(x+6)(2x-7) = \\
&(-6+6)(2(-6)-7) = \\
&(0)(-19) = 0
\end{aligned}
\qquad\qquad
\begin{aligned}
&Check: x = \frac{7}{2} \\
&(x+6)(2x-7) = \\
&\left(\frac{7}{2}+6\right)\left(2\cdot\frac{7}{2}-7\right) = \\
&\left(\frac{19}{2}\right)(7-7) = \\
&\left(\frac{19}{2}\right)(0) = 0
\end{aligned}$$

Both solutions check out.

Factoring a polynomial is very useful because the Zero-Product Property allows us to break up the problem into simpler separate steps. When we can't factor a polynomial, the problem becomes harder and we must use other methods that you will learn later.

As a last note in this section, keep in mind that the Zero-Product Property only works when a product equals zero. For example, if you multiplied two numbers and the answer was nine, that wouldn't mean that one or both of the numbers must be nine. In order to use the property, the factored polynomial must be equal to zero.

Example B

Solve each equation:

a) $(x-9)(3x+4) = 0$

b) $x(5x-4) = 0$

c) $4x(x+6)(4x-9) = 0$

Solution

Since all the polynomials are in factored form, we can just set each factor equal to zero and solve the simpler equations separately

a) $(x-9)(3x+4) = 0$ can be split up into two linear equations:

$$\begin{aligned}
&x-9 = 0 \\
&\underline{\underline{x = 9}}
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
&3x+4 = 0 \\
&3x = -4 \\
&\underline{\underline{x = -\frac{4}{3}}}
\end{aligned}$$

b) $x(5x-4) = 0$ can be split up into two linear equations:

$$\underline{\underline{x = 0}}
\qquad \text{or} \qquad
\begin{aligned}
&5x-4 = 0 \\
&5x = 4 \\
&\underline{\underline{x = \frac{4}{5}}}
\end{aligned}$$

c) $4x(x+6)(4x-9) = 0$ can be split up into three linear equations:

$$4x = 0 \quad \text{or} \quad x+6=0 \quad \text{or} \quad 4x-9=0$$
$$x = \frac{0}{4} \qquad\qquad x=-6 \qquad\qquad 4x=9$$
$$\underline{\underline{x=0}} \qquad\qquad\qquad\qquad \underline{\underline{x=\frac{9}{4}}}$$

Solve Simple Polynomial Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-Product Property to solve polynomials in factored form—now we can use that knowledge to solve polynomials by factoring them first. Here are the steps:

a) If necessary, **rewrite** the equation in standard form so that the right-hand side equals zero.

b) **Factor** the polynomial completely.

c) Use the zero-product rule to **set each factor equal to zero.**

d) **Solve** each equation from step 3.

e) **Check** your answers by substituting your solutions into the original equation

Example C

Solve the following polynomial equations.

a) $x^2 - 2x = 0$

b) $2x^2 = 5x$

c) $9x^2y - 6xy = 0$

Solution

a) $x^2 - 2x = 0$

Rewrite: this is not necessary since the equation is in the correct form.

Factor: The common factor is x, so this factors as $x(x-2) = 0$.

Set each factor equal to zero:

$$x = 0 \qquad \text{or} \qquad x-2=0$$

Solve:

$$\underline{x=0} \qquad \text{or} \qquad \underline{x=2}$$

Check: Substitute each solution back into the original equation.

$$x=0 \Rightarrow (0)^2 - 2(0) = 0 \qquad \text{works out}$$
$$x=2 \Rightarrow (2)^2 - 2(2) = 4-4=0 \qquad \text{works out}$$

Answer: $x = 0, x = 2$

b) $2x^2 = 5x$

Rewrite: $2x^2 = 5x \Rightarrow 2x^2 - 5x = 0$

Factor: The common factor is x, so this factors as $x(2x-5) = 0$.

Set each factor equal to zero:

$$x = 0 \qquad \text{or} \qquad 2x - 5 = 0$$

Solve:

$$\underline{x = 0} \qquad \text{or} \qquad \begin{array}{l} 2x = 5 \\ \underline{x = \frac{5}{2}} \end{array}$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 \Rightarrow 2(0)^2 = 5(0) \Rightarrow 0 = 0 & \text{works out} \\ x = \frac{5}{2} \Rightarrow 2\left(\frac{5}{2}\right)^2 = 5 \cdot \frac{5}{2} \Rightarrow 2 \cdot \frac{25}{4} = \frac{25}{2} \Rightarrow \frac{25}{2} = \frac{25}{2} & \text{works out} \end{array}$$

Answer: $x = 0, x = \frac{5}{2}$

c) $9x^2y - 6xy = 0$

Rewrite: not necessary

Factor: The common factor is $3xy$, so this factors as $3xy(3x-2) = 0$.

Set each factor equal to zero:

$3 = 0$ is never true, so this part does not give a solution. The factors we have left give us:

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad 3x - 2 = 0$$

Solve:

$$\underline{x = 0} \qquad \text{or} \qquad \begin{array}{l} \underline{y = 0} \\ \underline{x = \frac{2}{3}} \end{array} \qquad \text{or} \qquad 3x = 2$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 \Rightarrow 9(0)y - 6(0)y = 0 - 0 = 0 & \text{works out} \\ y = 0 \Rightarrow 9x^2(0) - 6x(0) = 0 - 0 = 0 & \text{works out} \\ x = \frac{2}{3} \Rightarrow 9 \cdot \left(\frac{2}{3}\right)^2 y - 6 \cdot \frac{2}{3}y = 9 \cdot \frac{4}{9}y - 4y = 4y - 4y = 0 & \text{works out} \end{array}$$

Answer: $x = 0, y = 0, x = \frac{2}{3}$

Vocabulary

- Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:
- The **factored form** of a polynomial means it is written as a product of its factors.
- **Zero Product Property:** The only way a product is zero is if one or more of the terms are equal to zero:

$a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

Guided Practice

Solve the following polynomial equation.

$$9x^2 - 3x = 0$$

Solution: $9x^2 - 3x = 0$

Rewrite: This is not necessary since the equation is in the correct form.

Factor: The common factor is $3x$, so this factors as: $3x(3x-1) = 0$.

Set each factor equal to zero.

$$3x = 0 \qquad \text{or} \qquad x - 2 = 0$$

Solve:

$$x = 0 \qquad \text{or} \qquad x = 2$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 & (0)^2 - 2(0) = 0 \\ x = 2 & (2)^2 - 2(2) = 0 \end{array}$$

Answer $x = 0,\ x = 2$

Practice

Solve the following polynomial equations.

1. $x(x+12) = 0$
2. $(2x+1)(2x-1) = 0$
3. $(x-5)(2x+7)(3x-4) = 0$
4. $2x(x+9)(7x-20) = 0$

5. $x(3+y)=0$
6. $x(x-2y)=0$
7. $18y-3y^2=0$
8. $9x^2=27x$
9. $4a^2+a=0$
10. $b^2-\frac{5}{3}b=0$
11. $4x^2=36$
12. $x^3-5x^2=0$

7.10 Quadratic Formula

Here you'll learn how to use the quadratic formula to find the vertex and solution of quadratic equations.

What if you had a quadratic equation like x^2+5x+2 that you could not easily factor? How could you use its coeffient values to solve it? After completing this Concept, you'll be able to use the quadratic formula to solve equations like this one.

Guidance

The **Quadratic Formula** is probably the most used method for solving quadratic equations. For a quadratic equation in standard form, $ax^2+bx+c=0$, the quadratic formula looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation: $ax^2+bx+c=0$

Subtract the constant term from both sides: $ax^2+bx=-c$

Divide by the coefficient of the x^2 term:

$x^2+\frac{b}{a}x=-\frac{c}{a}$

Rewrite:

$x^2+2\left(\frac{b}{2a}\right)x=-\frac{c}{a}$

Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides:

$x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$

Factor the perfect square trinomial:

$\left(x+\frac{b}{2a}\right)^2=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}$

Simplify:

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Take the square root of both sides:

$x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{4a^2}}$ and $x+\frac{b}{2a}=-\sqrt{\frac{b^2-4ac}{4a^2}}$

Simplify:

$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a}$ and $x+\frac{b}{2a}=-\frac{\sqrt{b^2-4ac}}{2a}$

$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$ and $x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$

$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$

This can be written more compactly as $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious, so the quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

To use the quadratic formula, just plug in the values of a, b, and c.

Example A

Solve the following quadratic equations using the quadratic formula.

a) $2x^2+3x+1=0$

b) $x^2-6x+5=0$

c) $-4x^2+x+1=0$

Solution

Start with the quadratic formula and plug in the values of a, b and c.

a)

$$\begin{aligned}
\text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\
\text{Plug in the values } a=2,\ b=3,\ c=1 \quad & x=\frac{-3\pm\sqrt{(3)^2-4(2)(1)}}{2(2)} \\
\text{Simplify:} \quad & x=\frac{-3\pm\sqrt{9-8}}{4}=\frac{-3\pm\sqrt{1}}{4} \\
\text{Separate the two options:} \quad & x=\frac{-3+1}{4} \text{ and } x=\frac{-3-1}{4} \\
\text{Solve:} \quad & x=\frac{-2}{4}=-\frac{1}{2} \text{ and } x=\frac{-4}{4}=-1
\end{aligned}$$

Answer: $x=-\frac{1}{2}$ and $x=-1$

b)

$$\begin{aligned}
\text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\
\text{Plug in the values } a=1,\ b=-6,\ c=5 \quad & x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(5)}}{2(1)} \\
\text{Simplify:} \quad & x=\frac{6\pm\sqrt{36-20}}{2}=\frac{6\pm\sqrt{16}}{2} \\
\text{Separate the two options:} \quad & x=\frac{6+4}{2} \text{ and } x=\frac{6-4}{2} \\
\text{Solve:} \quad & x=\frac{10}{2}=5 \text{ and } x=\frac{2}{2}=1
\end{aligned}$$

Answer: $x=5$ and $x=1$

c)

$$\begin{aligned} \text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ \text{Plug in the values } a=-4,\ b=1,\ c=1 \quad & x=\frac{-1\pm\sqrt{(1)^2-4(-4)(1)}}{2(-4)} \\ \text{Simplify:} \quad & x=\frac{-1\pm\sqrt{1+16}}{-8}=\frac{-1\pm\sqrt{17}}{-8} \\ \text{Separate the two options:} \quad & x=\frac{-1+\sqrt{17}}{-8} \text{ and } x=\frac{-1-\sqrt{17}}{-8} \\ \text{Solve:} \quad & x=-.39 \text{ and } x=.64 \end{aligned}$$

Answer: $x=-.39$ and $x=.64$

Often when we plug the values of the coefficients into the quadratic formula, we end up with a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced math classes, you'll learn how to work with "complex" (or "imaginary") solutions to quadratic equations.

Example B

Use the quadratic formula to solve the equation $x^2+2x+7=0$.

Solution

$$\begin{aligned} \text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ \text{Plug in the values } a=1,\ b=2,\ c=7 \quad & x=\frac{-2\pm\sqrt{(2)^2-4(1)(7)}}{2(1)} \\ \text{Simplify:} \quad & x=\frac{-2\pm\sqrt{4-28}}{2}=\frac{-2\pm\sqrt{-24}}{2} \end{aligned}$$

Answer: There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, that means we have to start by rewriting the equation.

Finding the Vertex of a Parabola with the Quadratic Formula

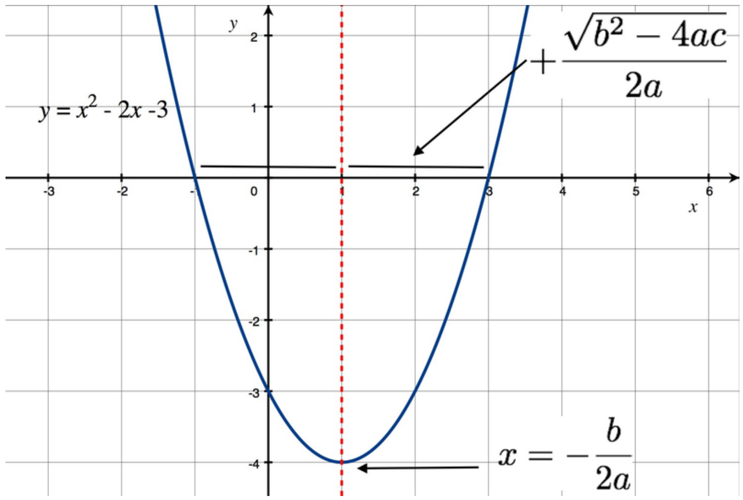
Sometimes a formula gives you even more information than you were looking for. For example, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

Remember that the quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$,, and we can rewrite that as $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$.

Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x=\frac{-b}{2a}$.

Example C

In the equation $x^2-2x-3=0$, the roots -1 and 3 are both 2 units from the vertical line $x=1$, as you can see in the graph below:



Vocabulary

- For a **quadratic equation** in standard form, $ax^2+bx+c=0$, the **quadratic formula** looks like this:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- The quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, and we can rewrite that as $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$.
- The roots are symmetric about the **vertex**. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x = \frac{-b}{2a}$.

Guided Practice

Solve the following equations using the quadratic formula.

a) $x^2 - 6x = 10$

b) $-8x^2 = 5x + 6$

Solution

a)

Re-write the equation in standard form: $x^2 - 6x - 10 = 0$

Quadratic formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Plug in the values $a = 1,\ b = -6,\ c = -10$ $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-10)}}{2(1)}$

Simplify: $x = \frac{6 \pm \sqrt{36+40}}{2} = \frac{6 \pm \sqrt{76}}{2}$

Separate the two options: $x = \frac{6+\sqrt{76}}{2}$ and $x = \frac{6-\sqrt{76}}{2}$

Solve: $x = 7.36$ and $x = -1.36$

Answer: $x = 7.36$ and $x = -1.36$

b)

Re-write the equation in standard form:	$8x^2 + 5x + 6 = 0$
Quadratic formula:	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Plug in the values $a = 8,\ b = 5,\ c = 6$	$x = \frac{-5 \pm \sqrt{(5)^2 - 4(8)(6)}}{2(8)}$
Simplify:	$x = \frac{-5 \pm \sqrt{25 - 192}}{16} = \frac{-5 \pm \sqrt{-167}}{16}$

Answer: no real solutions

Practice

Solve the following quadratic equations using the quadratic formula.

1. $x^2 + 4x - 21 = 0$
2. $x^2 - 6x = 12$
3. $3x^2 - \frac{1}{2}x = \frac{3}{8}$
4. $2x^2 + x - 3 = 0$
5. $-x^2 - 7x + 12 = 0$
6. $-3x^2 + 5x = 2$
7. $4x^2 = x$
8. $x^2 + 2x + 6 = 0$
9. $5x^2 - 2x + 100 = 0$
10. $100x^2 + 10x + 70 = 0$

7.11 Vertex, Intercept, and Standard Form

Here you'll explore the different forms of the quadratic equation.

The profit on your school fundraiser is represented by the quadratic expression $-5p^2+400p-8000$, where p is your price point. What price point will result in a maximum profit and what is that profit?

Guidance

So far, we have only used the **standard form** of a quadratic equation, $y=ax^2+bx+c$ to graph a parabola. From standard form, we can find the vertex and either factor or use the Quadratic Formula to find the $x-$intercepts. The **intercept form** of a quadratic equation is $y=a(x-p)(x-q)$, where a is the same value as in standard form, and p and q are the $x-$intercepts. This form looks very similar to a factored quadratic equation.

Example A

Change $y=2x^2+9x+10$ to intercept form and find the vertex. Graph the parabola.

Solution: First, let's change this equation into intercept form by factoring. $ac=20$ and the factors of 20 that add up to 9 are 4 and 5. Expand the $x-$term.

$$\begin{aligned}
&y=2x^2+9x+10\\
&y=2x^2+4x+5x+10\\
&y=2x(x+2)+5(x+2)\\
&y=(2x+5)(x+2)
\end{aligned}$$

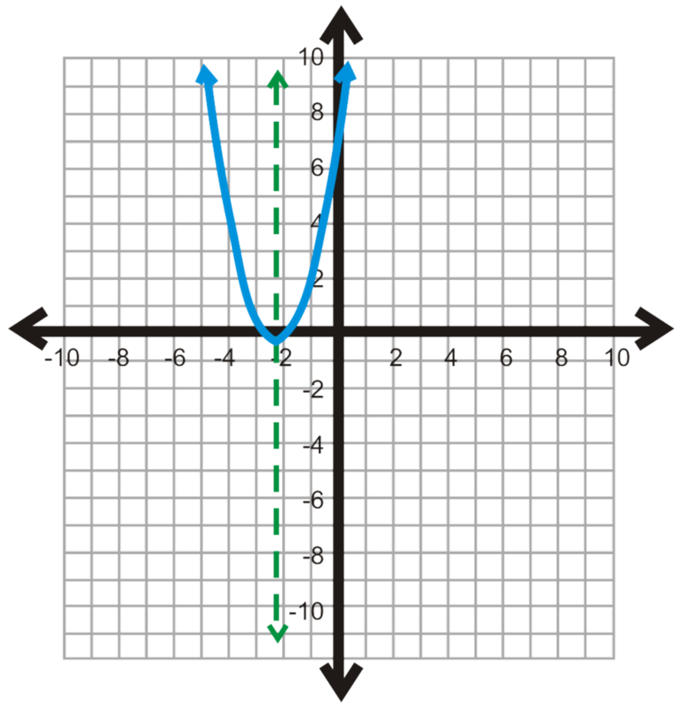
Notice, this does not exactly look like the definition. The factors cannot have a number in front of x. Pull out the 2 from the first factor to get $y=2\left(x+\frac{5}{2}\right)(x+2)$. Now, find the vertex. Recall that all parabolas are symmetrical. This means that the axis of symmetry is *halfway* between the $x-$intercepts or their average.

$$\text{axis of symmetry}=\frac{p+q}{2}=\frac{-\frac{5}{2}-2}{2}=-\frac{9}{2}\div 2=-\frac{9}{2}\cdot\frac{1}{2}=-\frac{9}{4}$$

This is also the $x-$coordinate of the vertex. To find the $y-$coordinate, plug the $x-$value into either form of the quadratic equation. We will use Intercept form.

$$\begin{aligned}
&y=2\left(-\frac{9}{4}+\frac{5}{2}\right)\left(-\frac{9}{4}+2\right)\\
&y=2\cdot\frac{1}{4}\cdot-\frac{1}{4}\\
&y=-\frac{1}{8}
\end{aligned}$$

The vertex is $\left(-2\frac{1}{4},-\frac{1}{8}\right)$. Plot the $x-$intercepts and the vertex to graph.

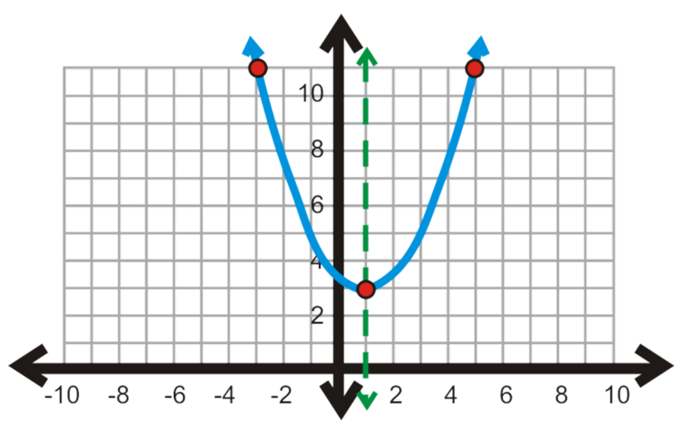


The last form is vertex form. **Vertex form** is written $y = a(x-h)^2 + k$, where (h,k) is the vertex and a is the same is in the other two forms. Notice that h is negative in the equation, but positive when written in coordinates of the vertex.

Example B

Find the vertex of $y = \frac{1}{2}(x-1)^2 + 3$ and graph the parabola.

Solution: The vertex is going to be (1, 3). To graph this parabola, use the symmetric properties of the function. Pick a value on the left side of the vertex. If $x = -3$, then $y = \frac{1}{2}(-3-1)^2 + 3 = 11$. -3 is 4 units away from 1 (the $x-$coordinate of the vertex). 4 units on the *other* side of 1 is 5. Therefore, the $y-$coordinate will be 11. Plot (1, 3), (-3, 11), and (5, 11) to graph the parabola.



Example C

Change $y = x^2 - 10x + 16$ into vertex form.

Solution: To change an equation from standard form into vertex form, you must complete the square. Review the *Completing the Square* Lesson if needed. The major difference is that you will not need to solve this equation.

$$\begin{aligned} y &= x^2 - 10x + 16 \\ y - 16 + 25 &= x^2 - 10x + 25 && \text{Move 16 to the other side and add } \left(\frac{b}{2}\right)^2 \text{ to both sides.} \\ y + 9 &= (x-5)^2 && \text{Simplify left side and factor the right side} \\ y &= (x-5)^2 - 9 && \text{Subtract 9 from both sides to get } y \text{ by itself.} \end{aligned}$$

To solve an equation in vertex form, set $y = 0$ and solve for x.

$$\begin{aligned} (x-5)^2 - 9 &= 0 \\ (x-5)^2 &= 9 \\ x - 5 &= \pm 3 \\ x &= 5 \pm 3 \textit{ or } 8 \textit{ and } 2 \end{aligned}$$

Intro Problem Revisit The vertex will give us the price point that will result in the maximum profit and that profit, so let's change this equation into intercept form by factoring. First factor out –5.

$$\begin{aligned} -5p^2 + 400p - 8000 &= -5(p^2 - 80p + 1600) \\ &-5(p-40)(p-40) \end{aligned}$$

From this we can see that the x-intercepts are 40 and 40. The average of 40 and 40 is 40 we plug 40 into our original equation.

$$-5(40)^2 + 400(40) - 8000 = -8000 + 16000 - 8000 = 0$$

Therefore, the price point that results in a maximum profit is \$40 and that price point results in a profit of \$0. You're not making any money, so you better rethink your fundraising approach!

Guided Practice

1. Find the intercepts of $y = 2(x-7)(x+2)$ and change it to standard form.
2. Find the vertex of $y = -\frac{1}{2}(x+4)^2 - 5$ and change it to standard form.
3. Change $y = x^2 + 18x + 45$ to intercept form and graph.
4. Change $y = x^2 - 6x - 7$ to vertex form and graph.

Answers

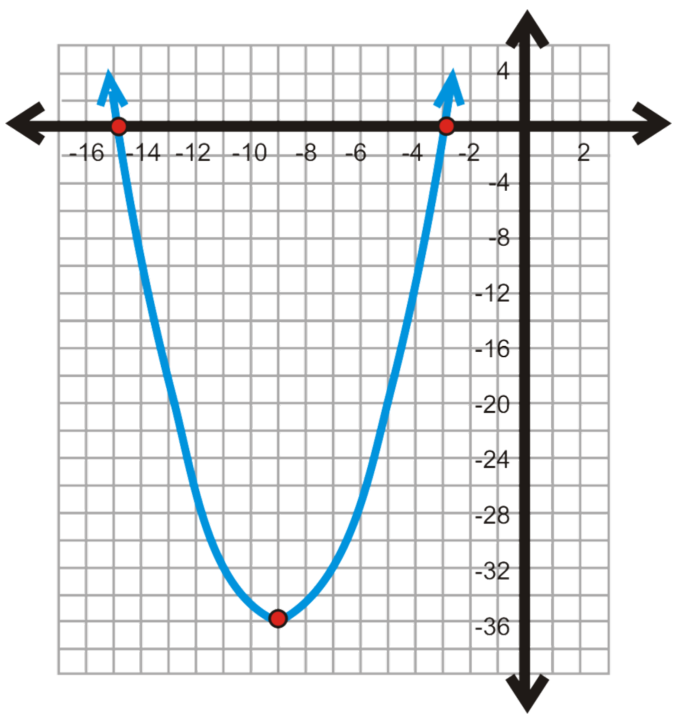
1. The intercepts are the opposite sign from the factors; (7, 0) and (-2, 0). To change the equation into standard form, FOIL the factors and distribute a.

$$y = 2(x-7)(x+2)$$
$$y = 2(x^2 - 5x - 14)$$
$$y = 2x^2 - 10x - 28$$

2. The vertex is (-4, -5). To change the equation into standard form, FOIL $(x+4)^2$, distribute a, and then subtract 5.

$$y = -\frac{1}{2}(x+4)(x+4) - 5$$
$$y = -\frac{1}{2}(x^2 + 8x + 16) - 5$$
$$y = -\frac{1}{2}x^2 - 4x - 21$$

3. To change $y = x^2 + 18x + 45$ into intercept form, factor the equation. The factors of 45 that add up to 18 are 15 and 3. Intercept form would be $y = (x+15)(x+3)$. The intercepts are (-15, 0) and (-3, 0). The $x-$coordinate of the vertex is halfway between -15 and -3, or -9. The $y-$coordinate of the vertex is $y = (-9)^2 + 18(-9) + 45 = -36$. Here is the graph:

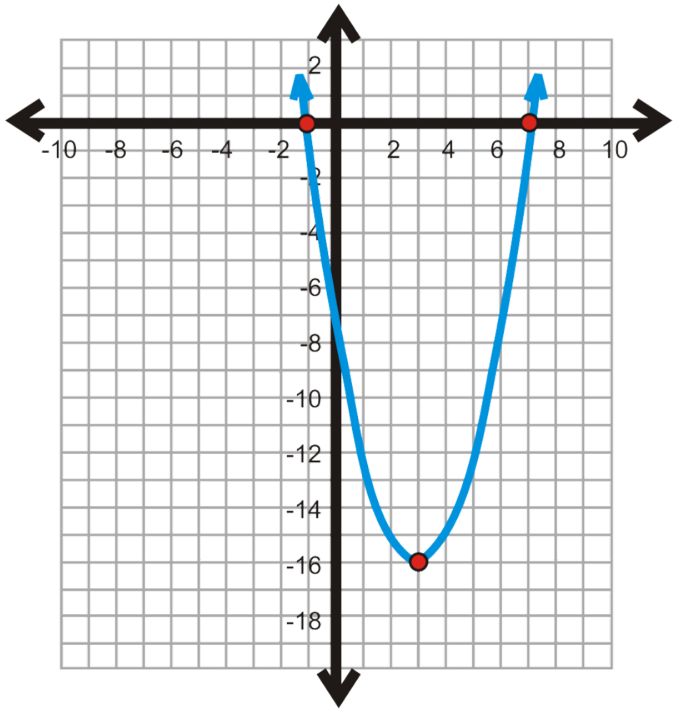


4. To change $y = x^2 - 6x - 7$ into vertex form, complete the square.

$$y + 7 + 9 = x^2 - 6x + 9$$
$$y + 16 = (x-3)^2$$
$$y = (x-3)^2 - 16$$

The vertex is (3, -16).

For vertex form, we could solve the equation by using square roots or we could factor the standard form. Either way, we will get that the $x-$intercepts are (7, 0) and (-1, 0).



Vocabulary

Standard form

$y = ax^2 + bx + c$

Intercept form

$y = a(x-p)(x-q)$, where p and q are the $x-$intercepts.

Vertex form

$y = a(x-h)^2 + k$, where (h,k) is the vertex.

Practice

1. Fill in the table below. Either describe how to find each entry or use a formula.

	Equation	Vertex	Intercepts
Standard Form			
Intercept Form			
Vertex Form			

Find the vertex and $x-$intercepts of each function below. Then, graph the function. If a function does not have any $x-$intercepts, use the symmetry property of parabolas to find points on the graph.

2. $y = (x-4)^2 - 9$

3. $y = (x+6)(x-8)$
4. $y = x^2 + 2x - 8$
5. $y = -(x-5)(x+7)$
6. $y = 2(x+1)^2 - 3$
7. $y = 3(x-2)^2 + 4$
8. $y = \frac{1}{3}(x-9)(x+3)$
9. $y = -(x+2)^2 + 7$
10. $y = 4x^2 - 13x - 12$

Change the following equations to intercept form.

11. $y = x^2 - 3x + 2$
12. $y = -x^2 - 10x + 24$
13. $y = 4x^2 + 18x + 8$

Change the following equations to vertex form.

14. $y = x^2 + 12x - 28$
15. $y = -x^2 - 10x + 24$
16. $y = 2x^2 - 8x + 15$

Change the following equations to standard form.

17. $y = (x-3)^2 + 8$
18. $y = 2\left(x - \frac{3}{2}\right)(x-4)$
19. $y = -\frac{1}{2}(x+6)^2 - 11$