1.7 Square Roots and Real Numbers

Learning Objectives

- Find square roots.
- Approximate square roots.
- Identify irrational numbers.
- Classify real numbers.
- Graph and order real numbers.

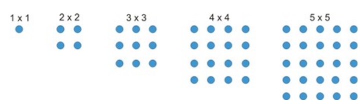
Find Square Roots

The square root of a number is a number which, when multiplied by itself gives the original number. In algebraic terms, the square root of x is a number, b, such that $b^2 = x$.

Note: There are two possibilities for a numerical value for b. The **positive** number that satisfies the equation $b^2 = x$ is called the **principal square root**. Since $(-b) \cdot (-b) = +b^2 = x, -b$ is also a valid solution.

The square root of a number, x, is written as $\sqrt{x}$ or sometimes as $\sqrt[2]{x}$. For example, $2^2 = 4$, so the square root of 4, $\sqrt{4} = \pm 2$.

Some numbers, like 4, have integer square roots. Numbers with integer square roots are called **perfect squares**. The first five perfect squares $(1, 4, 9, 16, 25)$ are shown below.



You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. Further, to find the square root of that number, simply take one of each pair of factors and multiply them together.

Example 1

Find the principal square root of each of these perfect squares.

a) 121

b) 225

c) 324

d) 576

a) $121 = 11 \times 11$

Solution

$$\sqrt{121} = 11$$

b) $225 = (5 \times 5) \times (3 \times 3)$

Solution

$$\sqrt{225} = 5 \times 3 = 15$$

c) $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$

Solution

$$\sqrt{324} = 2 \times 3 \times 3 = 18$$

d) $576 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (3 \times 3)$

Solution

$$\sqrt{576} = 2 \times 2 \times 2 \times 3 = 24$$

Approximate Square Roots

When we have perfect squares, we can write an exact numerical solution for the principal square root. When we have one or more unpaired primes in the factor tree of a number, however, we do not get integer values for the square root and we have seen that we leave a radical in the answer. Terms like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{\ }$ or $\sqrt{x}$ button on a calculator. When the number we are finding the square root of is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the decimals will appear random and we will have an irrational number as our answer. We call this an **approximate answer.** Even though we may have an answer to eight or nine decimal places, it still represents an **approximation** of the real answer which has an **infinite number of non-repeating decimals**.

Example 4

Use a calculator to find the following square roots. Round your answer to three decimal places.

a) $\sqrt{99}$

b) $\sqrt{5}$

c) $\sqrt{0.5}$

d) $\sqrt{1.75}$

a) The calculator returns 9.949874371.

Solution

$$\sqrt{99} \approx 9.950$$

b) The calculator returns 2.236067977.

Solution

$$\sqrt{5} \approx 2.236$$

c) The calculator returns 0.7071067812 .

Solution

$$\sqrt{0.5} \approx 0.707$$

d) The calculator returns 1.322875656.

Solution

$$\sqrt{1.75} \approx 1.323$$

Identify Irrational Numbers

Any square root that cannot be simplified to a form without a square root is **irrational,** but **not all** square roots are irrational. For example, $\sqrt{49}$ = 7. This is a terminating decimal, which makes $\sqrt{49}$ is **rational**, but $\sqrt{50}$ does not simplify perfectly. $\sqrt{50}\approx$ 7.071067812. The fact that it is a non-terminating non-repeating decimal makes $\sqrt{50}$**irrational**.

Example 5

Identify which of the following are rational numbers and which are irrational numbers.

a) 23.7

b) 2.8956

c) π

d) $\sqrt{6}$

e) $3.\overline{27}$

a) $23.7 = 23\frac{7}{10}$. This is clearly a **rational number**.

b) $2.8956 = 2\frac{8956}{10000}$. Again, this is a **rational number**.

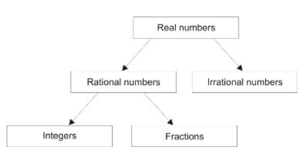
c) $\pi = 3.141592654\ldots$ The decimals appear random, and from the definition of π we know they do not repeat. This is an **irrational number**.

d) $\sqrt{6} = 2.44949489743\ldots$ Again the decimals appear to be random. We also know that $\sqrt{6} = \sqrt{2} \times \sqrt{3}$. Square roots of **primes** are irrational. $\sqrt{6}$ is an irrational number.

e) $3.\overline{27} = 3.2727272727\ldots$ Although these decimals are recurring they are certainly *not* unpredictable. This is a **rational number** (in actual fact, $3.\overline{27} = \frac{36}{11}$)

Classify Real Numbers

We can now see how real numbers fall into one of several categories.



If a real number can be expressed as a rational number, it falls into one of two categories. If the denominator of its **simplest form** is one, then it is an **integer**. If not, it is a fraction (this term is used here to also include decimals, as $3.27 = 3\frac{27}{100}$. **Rational numbers will always be terminating decimals or non-terminating repeating decimals.**

If the number cannot be expressed as the ratio of two integers (i.e. as a fraction), it is **irrational**. **Irrational numbers will always be non-terminating, non-repeating decimals.**

Example 6

Classify the following real numbers.

a) 0

b) -1

c) $\frac{\pi}{3}$

d) $\frac{\sqrt{2}}{3}$

e) $\frac{\sqrt{36}}{9}$

a) **Solution**

Zero is an **integer.**

b) **Solution**

-1 is an **integer**.

c) Although $\frac{\pi}{3}$ is written as a fraction, the numerator (π) is irrational.

Solution

$\frac{\pi}{3}$ is an **irrational number**.

d) $\frac{\sqrt{2}}{3}$ cannot be simplified to remove the square root.

Solution

$\frac{\sqrt{2}}{3}$ is an **irrational number**.

e) $\frac{\sqrt{36}}{9}$ can be simplified to $\frac{\sqrt{36}}{3} = \frac{6}{9} = \frac{2}{3}$**Solution**

$\frac{\sqrt{36}}{9}$ is a **rational number.**

Graph and Order Real Numbers

We have already talked about plotting integers on the number line. It gives a visual representation of which number is bigger, smaller, etc. It would therefore be helpful to plot non-integer rational numbers (fractions) on the number line also. There are two ways to graph rational numbers on the number line. You can convert them to a mixed number (graphing is one of the few instances in algebra when mixed numbers are preferred to improper fractions), or you can convert them to decimal form.

Example 7

Plot the following rational numbers on the number line.

a) $\frac{2}{3}$

b) $-\frac{3}{7}$

c) $\frac{17}{3}$

d) $\frac{57}{16}$

If we divide the intervals on the number line into the number on the denominator, we can look at the fraction's numerator to determine how many of these **sub-intervals** we need to include.