CHAPTER 1

Chapter 1: Real Numbers

Chapter Outline

1.1 THE REAL NUMBER SYSTEM
1.2 INTEGER OPERATIONS
1.3 ORDER OF OPERATIONS
1.4 DISTRIBUTIVE PROPERTY
1.5 SQUARE ROOTS AND IRRATIONAL NUMBERS

1.1 The Real Number System

Here you'll learn how to identify the subsets of real numbers and place a real number into one of these subsets.

There are many different ways to classify or name numbers.

All of the numbers we'll work with in this class are considered *real numbers.*

When you were in the lower grades, you worked with ***whole numbers.*** **Whole numbers are counting numbers. We consider whole numbers as the set of numbers** $\{0,1,2,3,4\ldots\}$.

Later, you may also have learned about ***integers***. **The set of integers includes whole numbers, but also includes their opposites.** Therefore, we can say that whole positive and negative numbers are part of the set of integers $\{\ldots-2,-1,0,1,2,3\ldots\}$.

We can't stop classifying numbers with whole numbers and integers because sometimes we can measure a part of a whole or a whole with parts. These numbers are called ***rational numbers.*** **A rational number is any number that can be written as a fraction where the numerator or the denominator is not equal to zero.** Let's think about this. A whole number or an integer could also be a rational number because we can put it over 1.

-4 could be written as $-\frac{4}{1}$, therefore it is an integer, but also a rational number.

Exactly. We can also think about decimals too. Many decimals can be written as fractions, so decimals are also rational numbers.

There are two special types of decimals that are considered rational numbers and one kind of decimal that is NOT a rational number. A ***terminating decimal*** is a decimal that is considered to be a rational number. **A terminating decimal is a decimal that looks like it goes on and on, but at some point has an end.** It terminates or ends somewhere.

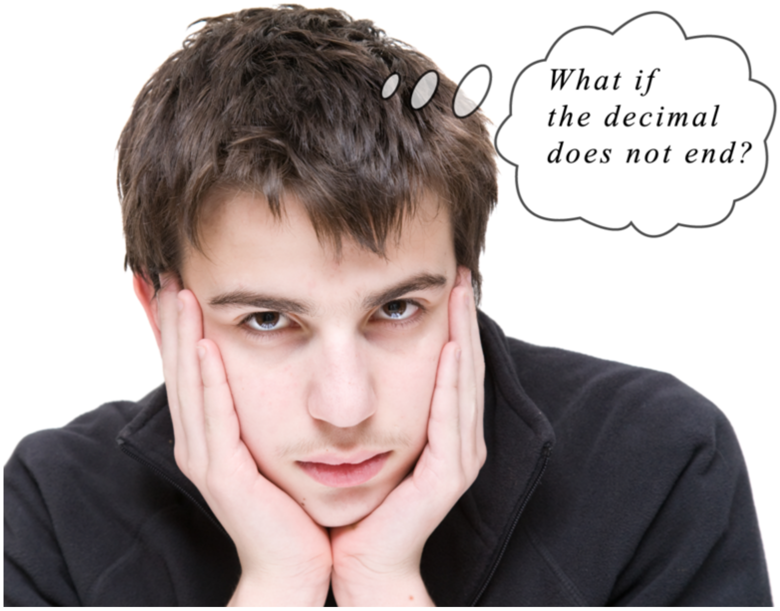
.3456798

This is a terminating decimal. It goes on for a while, but then ends.

A ***repeating decimal*** is also considered a rational number. **A repeating decimal has values that repeat forever.**

.676767679...

This is a repeating decimal.



Ah ha! This is the last type of number that is a decimal, but is NOT a rational number. It is called an *irrational number.* An irrational number is a decimal that does not end and has no repetition. It goes on and on and on. Irrational numbers cannot be represented as fractions. The most famous irrational number is *pi* (π). We use 3.14 to represent π, but you should know that *pi* is an irrational number meaning that it goes on and on and on forever.

How can we determine if a fraction or a decimal is rational or irrational?

If a number can be written in fraction form then it is rational. If a number cannot be written in fraction form then it is irrational. Besides π, roots of many numbers are also examples of irrational numbers. For example, $\sqrt{2}$ and $\sqrt{3}$ are both irrational numbers.

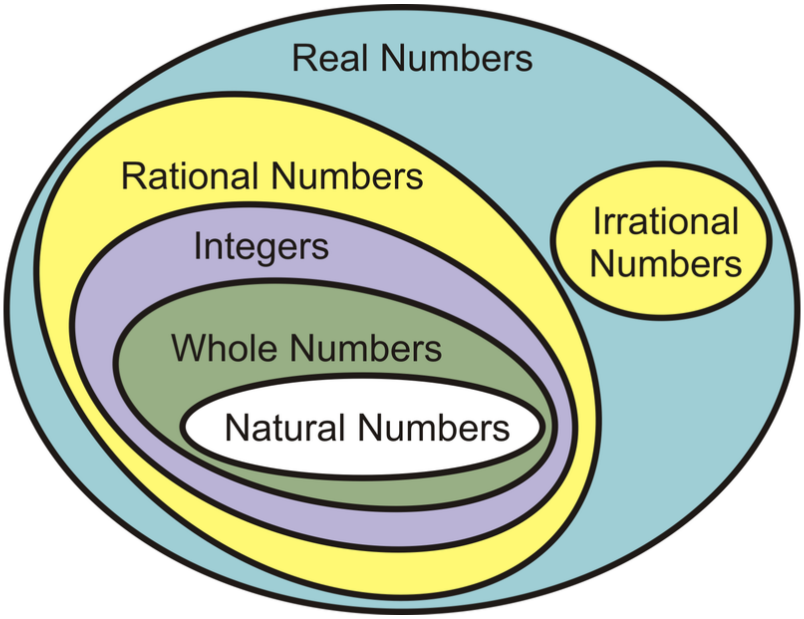
The table and diagram below summarize the real number system:

TABLE 1.1:

Real Numbers	Any number that can be plotted on a number line. *Symbol:* $\mathbb{R}$	*Examples:* $8, 4.67, -\frac{1}{3}, \pi$
Rational Numbers	Any number that can be written as a fraction, including repeating decimals. *Symbol:* $\mathbb{Q}$	*Examples:* $-\frac{5}{9}, \frac{1}{8}, 1.\overline{3}, \frac{16}{4}$
Irrational Numbers	Real numbers that are not rational. When written as a decimal, these numbers do not end nor repeat.	*Example:* $e, \pi, -\sqrt{2}, \sqrt[3]{5}$
Integers	All positive and negative "counting" numbers and zero. *Symbol:* $\mathbb{Z}$	*Example:* -4, 6, 23, -10
Whole Numbers	All positive "counting" numbers and zero.	*Example:* 0, 1, 2, 3, ...
Natural Numbers	All positive "counting" numbers. *Symbol:* $\mathbb{N}$	*Example:* 1, 2, 3, ...

A **counting number** is any number that can be counted on your fingers.

The real numbers can be grouped together as follows:



Classifying Real Numbers

Classify each real number.

Example A

$\sqrt{7}$

Solution: Irrational number

Example B

$\frac{1}{9}$

Solution: Rational number

Example C

-98

Solution: Integer and rational number

Writing a Repeating Decimal as a Fraction

Example: How do we write 0.14141414.... as a fraction? Let's devise a step-by-step process.

Step 1: Set your repeating decimal equal to x.

$$x = 0.14141414$$

Step 2: Find the repeating digit(s).

In this case 14 is repeating.

Step 3: Move the repeating digits to the left of the decimal point and leave the remaining digits to the right.

$$14.14141414$$

Step 4: Multiply x by the same factor you mulitplied your original repeating decimal to get your new repeating decimal.

$$14.14141414 = 100(0.14141414)$$

So,

$$100x = 14.14141414$$

Step 5: Solve your system of linear equations for x.

$$(100x = 14.14141414) - (x = 0.14141414)$$

yields:

$$99x = 14$$

, so $x = \frac{14}{99}$

What about 0.327272727... ? The 0.3 does not repeat. So, rewrite this as 0.727272727... – 0.4 Therefore, the fraction will be:

$$\begin{aligned} \frac{72}{99} - \frac{4}{10} \\ \frac{8}{11} - \frac{2}{5} \\ \frac{40}{55} - \frac{22}{55} \\ \frac{18}{55} \end{aligned}$$

Vocabulary

Subset
A set of numbers that is contained in a larger group of numbers.

Real Numbers
Any number that can be plotted on a number line.

Rational Numbers
Any number that can be written as a fraction, including repeating decimals.

Irrational Numbers
Real numbers that are not rational. When written as a decimal, these numbers do not end nor repeat.

Integers
All positive and negative "counting" numbers and zero.

Whole Numbers
All positive "counting" numbers and zero.

Natural Numbers or Counting Numbers
Numbers than can be counted on your fingers; 1, 2, 3, 4, ...

Terminating Decimal
When a decimal number ends.

Repeating Decimal
When a decimal number repeats itself in a pattern. 1.666..., 0.98989898... are examples of repeating decimals.

Guided Practice

1. What type of real number is $\sqrt{5}$?

2. List all the subsets that -8 is a part of.

3. True or False: $-\sqrt{9}$ is an irrational number.

Answers

1. $\sqrt{5}$ is an irrational number because, when converted to a decimal, it does not end nor does it repeat.

2. -8 is a negative integer. Therefore, it is also a rational number and a real number.

3. $-\sqrt{9} = -3$, which is an integer. The statement is false.

Practice

Directions: Classify each of the following numbers as real, whole, integer, rational or irrational. Some numbers will have more than one classification.

1. 3.45
2. -9
3. 1,270
4. 1.232323
5. $\frac{4}{5}$
6. -232,323
7. -98
8. 1.98
9. $\sqrt{16}$
10. $\sqrt{2}$

Directions: Answer each question as true or false.

11. An irrational number can also be a real number.
12. An irrational number is a real number and an integer.
13. A whole number is also an integer.
14. A decimal is considered a real number and a rational number.
15. A negative decimal can still be considered an integer.
16. An irrational number is a terminating decimal.
17. A radical is always an irrational number.
18. Negative whole numbers are integers and are also rational numbers.
19. Pi is an example of an irrational number.
20. A repeating decimal is also a rational number.

Rewrite the following repeating decimals as fractions.

21. 0.4646464646...
22. 0.81212121212...

1.2 Integer Operations

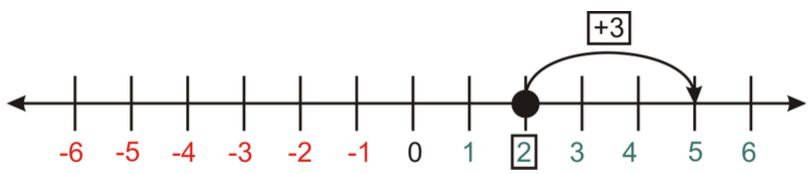
Here you'll learn the Commutative Property of Addition, Associative Property of Addition, and Identity Property of Addition so that you can effectively add integers.

Addition of Integers

A football team gains 11 yards on one play, then loses 5 yards on the next play, and then loses 2 yards on the third play. What is the total loss or gain of yardage?

A loss can be expressed as a negative integer. A gain can be expressed as a positive integer. To find the net gain or loss, the individual values must be added together. Therefore, the sum is $11+(-5)+(-2)=4$. The team has a net gain of 4 yards.

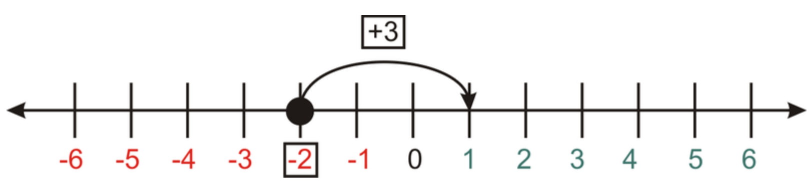
Addition can also be shown using a number line. If you need to add $2+3$, start by making a point at the value of 2 and move three integers to the right. The ending value represents the sum of the values.



Example A

Find the sum of $-2+3$ *using a number line.*

Solution: Begin by making a point at –2 and moving three units to the right. The final value is 1, so $-2+3=1$.

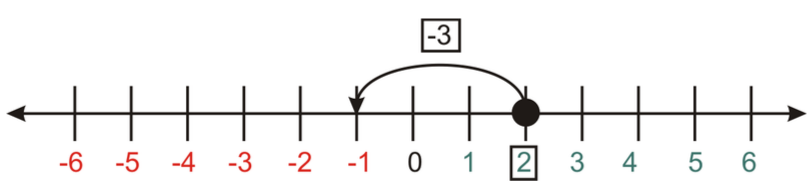


When the value that is being added is positive, we jump to the right. If the value is negative, we jump to the left (in a negative direction).

Example B

Find the sum of $2-3$ *using a number line.*

Solution: Begin by making a point at 2. The expression represents subtraction, so we will count three jumps to the left.



The solution is: $2-3=-1$.

Subtraction of Integers

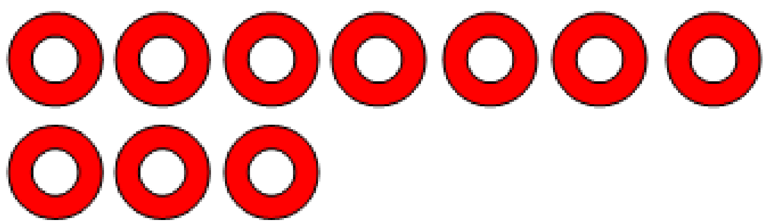
To subtract one signed number from another, change the problem from a subtraction problem to an addition problem and change the sign of the number that was originally being subtracted. In other words, to subtract signed numbers simply add the opposite. Then, follow the rules for adding signed numbers.

The subtraction of integers can be represented with manipulatives such as color counters and algebra tiles. A number line can also be used to show the subtraction of integers.

Example A

$7-(-3)=?$

Solution: This is the same as $7+(+3)=?$. The problem can be represented with color counters. In this case, the red counters represent positive numbers.



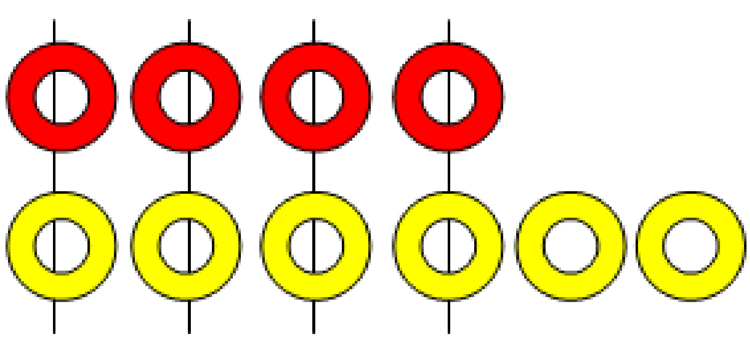
The answer is the sum of 7 and 3. $7+(+3)=10$

Example B

$4-(+6)=$

Solution: Change the problem to an addition problem and change the sign of the original number that was being subtracted.

$4-(+6)=4+(-6)=?$



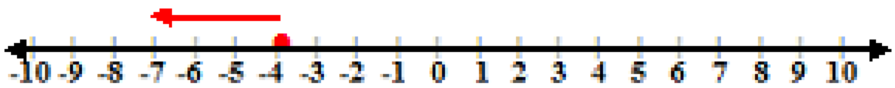
The remaining counters represent the answer. Therefore, $4-(+6)=-2$. The answer is the difference between 6 and 4 and takes the sign of the larger number.

Example C

$(-4)-(+3)=?$

Solution: This is the same as $(-4)+(-3)=?$. The solution to this problem can be determined by using the number line.

Indicate the starting point of -4 by using a dot. From this point, add a -3 by moving three places to the left. You will stop at -7.



The point where you stopped is the answer to the problem. Therefore, $(-4)-(+3)=-7$

Multiplication and Division of Integers

When you multiply and divide integers, there are some rules that need to be committed to memory.

- When multiplying or dividing a positive integer by a positive integer, the product or quotient is positive.
- When multiplying or dividing a positive integer by a negative integer, or a negative integer by a positive integer, the product or quotient is negative.
- When multiplying or dividing a negative integer by a negative integer, the product or quotient is positive.

You can remember these rules in a quicker way: if the signs are the same, the answer is positive. If the signs are different, the answer is negative.

Now let's apply these rules.

$12(-5)$

Notice that this is a multiplication problem. We use the parentheses around a single value to show multiplication. Now we can multiply the two values and then add the sign.

$12 \times 5 = 60$

A negative value times a positive value is a negative value.

Our answer is -60.

Here is another one.

$\frac{-150}{-50}$

Notice that this is a division problem. We use the fraction bar to show division. We do the division itself first.

$150 \div 50 = 3$

A negative divided by a negative is a positive.

The answer is 3.

Example A

$-9(7)$

Solution: -63

Example B

$-3(-12)$

Solution: 36

Example C

$-169 \div 13$

Solution: -13

Practice

Subtract.

1. $(-9)-(-2)$
2. $(5)-(+8)$
3. $(5)-(-4)$
4. $(-7)-(-9)$
5. $(6)-(+5)$
6. $(8)-(+4)$
7. $(-2)-(-7)$
8. $(3)-(+5)$
9. $(-6)-(-10)$
10. $(-4)-(-7)$
11. $(-13)-(-19)$
12. $(-6)-(+8)-(-12)$
13. $(14)-(+8)-(-6)$
14. $(18)-(+8)-(+3)$
15. $(10)-(-6)-(+4)-(+2)$

Directions: Multiply the following integers.

1. $-6(-8) =$ _____
2. $5(-10) =$ _____
3. $3(-4) =$ _____
4. $-3(4) =$ _____
5. $8(-9) =$ _____
6. $-9(12) =$ _____
7. $8(-11) =$ _____
8. $(-5)(-9) =$ _____
9. $-7(-8) =$ _____
10. $(-12)(12) =$ _____

Directions: Divide the following integers.

11. $-12 \div 2 =$ _____
12. $-18 \div -6 =$ _____
13. $-24 \div 12 =$ _____
14. $-80 \div -4 =$ _____
15. $-60 \div -30 =$ _____
16. $\frac{28}{4} =$ _____
17. $\frac{-36}{4} =$ _____
18. $\frac{-45}{-9} =$ _____
19. $-75 \div 25 =$ _____
20. $-68 \div -2 =$ _____

1.3 Order of Operations

Introduction

Look at and evaluate the following expression:

$$2+4\times 7-1=?$$

How many different ways can we interpret this problem, and how many different answers could someone possibly find for it?

The *simplest* way to evaluate the expression is simply to start at the left and work your way across:

$$\begin{aligned}&2+4\times 7-1\\&=6\times 7-1\\&=42-1\\&=41\end{aligned}$$

This is the answer you would get if you entered the expression into an ordinary calculator. But if you entered the expression into a scientific calculator or a graphing calculator you would probably get 29 as the answer.

In mathematics, the order in which we perform the various **operations** (such as adding, multiplying, etc.) is important. In the expression above, the operation of **multiplication** takes precedence over **addition**, so we evaluate it first. Let's re-write the expression, but put the multiplication in brackets to show that it is to be evaluated first.

$$2+(4\times 7)-1=?$$

First evaluate the brackets: $4\times 7=28$. Our expression becomes:

$$2+(28)-1=?$$

When we have only addition and subtraction, we start at the left and work across:

$$\begin{aligned}&2+28-1\\&=30-1\\&=29\end{aligned}$$

Algebra students often use the word **"PEMDAS"** to help remember the order in which we evaluate the mathematical expressions: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction.

Order of Operations

1. Evaluate expressions within **P**arentheses (also all brackets [ ] and braces { }) first.
2. Evaluate all **E**xponents (terms such as 3^2 or x^3) next.
3. **M**ultiplication *and* **D**ivision is next - work from left to right completing **both** multiplication and division in the order that they appear.
4. Finally, evaluate **A**ddition *and* **S**ubtraction - work from left to right completing **both** addition and subtraction in the order that they appear.

Evaluate Algebraic Expressions with Grouping Symbols

The first step in the order of operations is called **parentheses**, but we include all **grouping symbols** in this step—not just parentheses (), but also square brackets [ ] and curly braces { }.

Example 1

Evaluate the following:

a) $4-7-11+2$

b) $4-(7-11)+2$

c) $4-[7-(11+2)]$

Each of these expressions has the same numbers and the same mathematical operations, in the same order. The placement of the various grouping symbols means, however, that we must evaluate everything in a different order each time. Let's look at how we evaluate each of these examples.

a) This expression doesn't have parentheses, exponents, multiplication, or division. **PEMDAS** states that we treat addition and subtraction as they appear, starting at the left and working right (it's NOT addition *then* subtraction).

$$\begin{aligned}4-7-11+2 &= -3-11+2\\ &= -14+2\\ &= -12\end{aligned}$$

b) This expression has parentheses, so we first evaluate $7-11=-4$. Remember that when we subtract a negative it is equivalent to adding a positive:

$$\begin{aligned}4-(7-11)+2 &= 4-(-4)+2\\ &= 8+2\\ &= 10\end{aligned}$$

c) An expression can contain any number of sets of parentheses. Sometimes expressions will have sets of parentheses **inside** other sets of parentheses. When faced with **nested parentheses**, start at the innermost parentheses and work outward.

Brackets may also be used to group expressions which already contain parentheses. This expression has both brackets and parentheses. We start with the innermost group: $11+2=13$. Then we complete the operation in the brackets.

$$\begin{aligned}4-[7-(11+2)] &= 4-[7-(13)]\\&= 4-[-6]\\&= 10\end{aligned}$$

Example 2

Evaluate the following:

a) $3 \times 5 - 7 \div 2$

b) $3 \times (5-7) \div 2$

c) $(3 \times 5) - (7 \div 2)$

a) There are no grouping symbols. **PEMDAS** dictates that we multiply and divide first, working from left to right: $3 \times 5 = 15$ and $7 \div 2 = 3.5$. (NOTE: It's not multiplication *then* division.) Next we subtract:

$$\begin{aligned}3 \times 5 - 7 \div 2 &= 15 - 3.5\\&= 11.5\end{aligned}$$

b) First, we evaluate the expression inside the parentheses: $5 - 7 = -2$. Then work from left to right:

$$\begin{aligned}3 \times (5-7) \div 2 &= 3 \times (-2) \div 2\\&= (-6) \div 2\\&= -3\end{aligned}$$

c) First, we evaluate the expressions inside parentheses: $3 \times 5 = 15$ and $7 \div 2 = 3.5$. Then work from left to right:

$$\begin{aligned}(3 \times 5) - (7 \div 2) &= 15 - 3.5\\&= 11.5\end{aligned}$$

Note that adding parentheses didn't change the expression in part c, but did make it easier to read. Parentheses can be used to change the order of operations in an expression, but they can also be used simply to make it easier to understand.

We can also use the order of operations to simplify an expression that has variables in it, after we substitute specific values for those variables.

Example 3

Use the order of operations to evaluate the following:

a) $2-(3x+2)$ when $x = 2$

b) $3y^2 + 2y + 1$ when $y = -3$

c) $2-(t-7)^2 \times (u^3 - v)$ when $t = 19,\ u = 4$, and $v = 2$

a) The first step is to substitute the value for x into the expression. We can put it in parentheses to clarify the resulting expression.

$$2-(3(2)+2)$$

(Note: 3(2) is the same as 3×2.)

Follow **PEMDAS** - first parentheses. Inside parentheses follow **PEMDAS** again.

$2-(3\times 2+2)=2-(6+2)$	Inside the parentheses, we multiply first.
$2-8=-6$	Next we add inside the parentheses, and finally we subtract.

b) The first step is to substitute the value for y into the expression.

$$3\times(-3)^2+2\times(-3)-1$$

Follow **PEMDAS**: we cannot simplify the expressions in parentheses, so exponents come next.

$3\times(-3)^2+2\times(-3)-1$	Evaluate exponents: $(-3)^2=9$
$=3\times 9+2\times(-3)-1$	Evaluate multiplication: $3\times 9=27;\ 2\times -3=-6$
$=27+(-6)-1$	Add and subtract in order from left to right.
$=27-6-1$	
$=20$	

c) The first step is to substitute the values for $t,\ u$, and v into the expression.

$$2-(19-7)^2\times(4^3-2)$$

Follow **PEMDAS**:

$2-(19-7)^2\times(4^3-2)$	Evaluate parentheses: $(19-7)=12;\ (4^3-2)=(64-2)=62$
$=2-12^2\times 62$	Evaluate exponents: $12^2=144$
$=2-144\times 62$	Multiply: $144\times 62=8928$
$=2-8928$	Subtract.
$=-8926$	

In parts (b) and (c) we left the parentheses around the negative numbers to clarify the problem. They did not affect the order of operations, but they did help avoid confusion when we were multiplying negative numbers.

Part (c) in the last example shows another interesting point. When we have an expression inside the parentheses, we use **PEMDAS** to determine the order in which we evaluate the contents.

Evaluate Algebraic Expressions with Fraction Bars

Fraction bars count as grouping symbols for **PEMDAS**, so we evaluate them in the first step of solving an expression. All numerators and all denominators can be treated as if they have invisible parentheses around them. When **real** parentheses are also present, remember that the innermost grouping symbols come first. If, for example, parentheses appear on a numerator, they would take precedence over the fraction bar. If the parentheses appear outside of the fraction, then the fraction bar takes precedence.

Example 4

Use the order of operations to evaluate the following expressions:

a) $\frac{z+3}{4}-1$ when $z=2$

b) $\left(\frac{a+2}{b+4}-1\right)+b$ when $a=3$ and $b=1$

a) We substitute the value for z into the expression.

$$\frac{2+3}{4}-1$$

Although this expression has no parentheses, the fraction bar is also a grouping symbol—it has the same effect as a set of parentheses. We can write in the "invisible parentheses" for clarity:

$$\frac{(2+3)}{4}-1$$

Using **PEMDAS**, we first evaluate the numerator:

$$\frac{5}{4}-1$$

We can convert $\frac{5}{4}$ to a mixed number:

$$\frac{5}{4}=1\frac{1}{4}$$

Then evaluate the expression:

$$\frac{5}{4}-1=1\frac{1}{4}-1=\frac{1}{4}$$

b) We substitute the values for a and b into the expression:

$$\left(\frac{3+2}{1+4}-1\right)+1$$

This expression has nested parentheses (remember the effect of the fraction bar). The innermost grouping symbol is provided by the fraction bar. We evaluate the numerator $(3+2)$ and denominator $(1+4)$ first.

$$\left(\frac{3+2}{1+4}-1\right)+1=\left(\frac{5}{5}-1\right)+1$$ Next we evaluate the inside of the parentheses. First we divide.

$$=(1-1)+1$$ Next we subtract.

$$=0+1=1$$

Additional Resources

For more practice, you can play an algebra game involving order of operations online at http://www.funbrain.com/algebra/index.html .

Review Questions

1. Use the order of operations to evaluate the following expressions.
   a. $8-(19-(2+5)-7)$
   b. $2+7\times 11-12\div 3$
   c. $(3+7)\div(7-12)$
   d. $\frac{2\cdot(3+(2-1))}{4-(6+2)}-(3-5)$
   e. $\frac{4+7(3)}{9-4}+\frac{12-3\cdot 2}{2}$
   f. $(4-1)^2+3^2\cdot 2$
   g. $\frac{(2^2+5)^2}{5^2-4^2}\div(2+1)$
2. Evaluate the following expressions involving variables.
   a. $\frac{jk}{j+k}$ when $j=6$ and $k=12$
   b. $2y^2$ when $x=1$ and $y=5$
   c. $3x^2+2x+1$ when $x=5$
   d. $(y^2-x)^2$ when $x=2$ and $y=1$
   e. $\frac{x+y^2}{y-x}$ when $x=2$ and $y=3$
3. Evaluate the following expressions involving variables.
   a. $\frac{4x}{9x^2-3x+1}$ when $x=2$
   b. $\frac{z^2}{x+y}+\frac{x^2}{x-y}$ when $x=1,\ y=-2$, and $z=4$
   c. $\frac{4xyz}{y^2-x^2}$ when $x=3,\ y=2$, and $z=5$
   d. $\frac{x^2-z^2}{xz-2x(z-x)}$ when $x=-1$ and $z=3$
4. Insert parentheses in each expression to make a true equation.
   a. $5-2\times 6-5+2=5$
   b. $12\div 4+10-3\times 3+7=11$
   c. $22-32-5\times 3-6=30$
   d. $12-8-4\times 5=-8$

1.4 Distributive Property

Here you'll learn to identify and apply the Distributive Property to evaluate numerical expressions.

Evaluating Expressions with Both Products and Sums/Differences

What does the word "evaluate" mean?

When we ***evaluate*** an expression, we figure out the value of that expression or the quantity of the expression.

When we evaluate expressions that have a product and a sum, we use a ***property*** called the Distributive Property.

What is the Distributive Property?

The Distributive Property is a property that is a true statement about how to multiply a number with a sum. Multiply the number outside the parentheses with each number inside the parentheses. Then figure out the sum of those products.

In other words, we distribute the number outside the parentheses with both of the values inside the parentheses and find the sum of those numbers.

Let's see how this works.

$4(3+2)$

To use the Distributive Property, we take the four and multiply it by both of the numbers inside the parentheses. Then we find the sum of those products.

$$\begin{aligned}&4(3)+4(2)\\&12+8\\&20\end{aligned}$$

Our answer is 20.

Here is another one.

$8(9+4)$

Multiply the eight times both of the numbers inside the parentheses. Then find the sum of the products.

$$\begin{aligned}&8(9)+8(4)\\&72+32\\&104\end{aligned}$$

Our answer is 104.

Now it is your turn. Evaluate these expressions using the Distributive Property.

Example A

$5(6+3)$

Solution: 45

Example B

$2(8+1)$

Solution: 18

Example C

$12(3+2)$

Solution: 60

Vocabulary

Numerical expression
: a number sentence that has at least two different operations in it.

Product
: the answer in a multiplication problem

Sum
: the answer in an addition problem

Property
: a rule that works for all numbers

Evaluate
: to find the quantity of values in an expression

The Distributive Property
: the property that involves taking the product of the sum of two numbers. Take the number outside the parentheses and multiply it by each term in the parentheses.

Guided Practice

Here is one for you to try on your own.

Use the distributive property to evaluate this expression.

$4(9+2)$

First, we can distribute the four and multiply it by each value in the parentheses. Then we can add.

$36+8=44$

This is our answer.

Practice

Directions: Evaluate each expression using the Distributive Property.

1. 4(3 + 6)
2. 5(2 + 8)
3. 9(12 + 11)
4. 7(8 + 9)
5. 8(7 + 6)
6. 5(12 + 8)
7. 7(9 + 4)
8. 11(2 + 9)
9. 12(12 + 4)
10. 12(9 + 8)
11. 10(9 + 7)
12. 13(2 + 3)
13. 14(8 + 6)
14. 14(9 + 4)
15. 15(5 + 7)

1.5 Square Roots and Irrational Numbers

Here you'll learn how to find and approximate square roots. You'll also learn how to simplify expressions involving square roots.

What if you had a number like 1000 and you wanted to find its square root? After completing this concept, you'll be able to find square roots like this one by hand and with a calculator.

Try This

You can also work out square roots by hand using a method similar to long division. (See the web page at http://www.homeschoolmath.net/teaching/square-root-algorithm.php for an explanation of this method.)

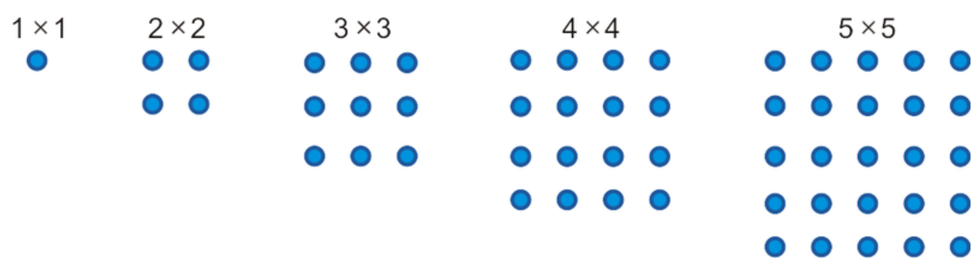
Guidance

The **square root** of a number is a number which, when multiplied by itself, gives the original number. In other words, if $a = b^2$, we say that b is the square root of a.

Note: Negative numbers and positive numbers both yield positive numbers when squared, so each positive number has both a positive and a negative square root. (For example, 3 and -3 can both be squared to yield 9.) The positive square root of a number is called the **principal square root**.

The square root of a number x is written as $\sqrt{x}$ or sometimes as $\sqrt[2]{x}$. The symbol $\sqrt{\ }$ is sometimes called a **radical sign**.

Numbers with whole-number square roots are called **perfect squares**. The first five perfect squares (1, 4, 9, 16, and 25) are shown below.



You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. To find the square root of that number, simply take one of each pair of matching factors and multiply them together.

Example A

Find the principal square root of each of these perfect squares.

a) 121

b) 225

c) 324

Solution

a) $121 = 11 \times 11$, so $\sqrt{121} = 11$.

b) $225 = (5 \times 5) \times (3 \times 3)$, so $\sqrt{225} = 5 \times 3 = 15$.

c) $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$, so $\sqrt{324} = 2 \times 3 \times 3 = 18$.

When the prime factors don't pair up neatly, we "factor out" the ones that do pair up and leave the rest under a radical sign. We write the answer as $a\sqrt{b}$, where a is the product of half the paired factors we pulled out and b is the product of the leftover factors.

Example B

Find the principal square root of the following numbers.

a) 8

b) 48

c) 75

Solution

a) $8 = 2 \times 2 \times 2$. This gives us one pair of 2's and one leftover 2, so $\sqrt{8} = 2\sqrt{2}$.

b) $48 = (2 \times 2) \times (2 \times 2) \times 3$, so $\sqrt{48} = 2 \times 2 \times \sqrt{3}$, or $4\sqrt{3}$.

c) $75 = (5 \times 5) \times 3$, so $\sqrt{75} = 5\sqrt{3}$.

Note that in the last example we collected the paired factors first, **then** we collected the unpaired ones under a single radical symbol. Here are the four rules that govern how we treat square roots.

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$

Example C

Simplify the following square root problems

a) $\sqrt{8} \times \sqrt{2}$

b) $3\sqrt{4} \times 4\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3}$

d) $12\sqrt{10} \div 6\sqrt{5}$

Solution

a) $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$

b) $3\sqrt{4} \times 4\sqrt{3} = 12\sqrt{12} = 12\sqrt{(2 \times 2) \times 3} = 12 \times 2\sqrt{3} = 24\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$

d) $12\sqrt{10} \div 6\sqrt{5} = \frac{12}{6}\sqrt{\frac{10}{5}} = 2\sqrt{2}$

Approximate Square Roots

Terms like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{\ }$ or $\sqrt{x}$ button on a calculator. When the number we plug in is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the answer will be irrational and will look like a random string of digits. Since the calculator can only show some of the infinitely many digits that are actually in the answer, it is really showing us an **approximate answer**—not exactly the right answer, but as close as it can get.

Example D

Use a calculator to find the following square roots. Round your answer to three decimal places.

a) $\sqrt{99}$

b) $\sqrt{5}$

c) $\sqrt{0.5}$

d) $\sqrt{1.75}$

Solution

a) ≈ 9.950

b) ≈ 2.236

c) ≈ 0.707

d) ≈ 1.323

Vocabulary

- The **square root** of a number is a number which gives the original number when multiplied by itself. In algebraic terms, for two numbers a and b, if $a = b^2$, then $b = \sqrt{a}$.
- A square root can have two possible values: a positive value called the **principal square root**, and a negative value (the opposite of the positive value).
- A **perfect square** is a number whose square root is an integer.
- Some mathematical properties of square roots are:
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
  - $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
  - $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
  - $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$
- Square roots of numbers that are not perfect squares (or ratios of perfect squares) are **irrational numbers**. They cannot be written as rational numbers (the ratio of two integers). In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.
- Computing a square root on a calculator will produce an **approximate solution** since the calculator only shows a finite number of digits after the decimal point.

Guided Practice

Find the square root of each number.

a) 576

b) 216

Solution

a) $576 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (3 \times 3)$, so $\sqrt{576} = 2 \times 2 \times 2 \times 3 = 24$.

b) $216 = (2 \times 2) \times 2 \times (3 \times 3) \times 3$, so $\sqrt{216} = 2 \times 3 \times \sqrt{2 \times 3}$, or $6\sqrt{6}$.

Practice

For 1-10, find the following square roots **exactly without using a calculator**, giving your answer in the simplest form.

1. $\sqrt{25}$
2. $\sqrt{24}$
3. $\sqrt{20}$
4. $\sqrt{200}$
5. $\sqrt{2000}$
6. $\sqrt{\frac{1}{4}}$ (Hint: The division rules you learned can be applied backwards!)
7. $\sqrt{\frac{9}{4}}$
8. $\sqrt{0.16}$
9. $\sqrt{0.1}$
10. $\sqrt{0.01}$

For 11-20, use a calculator to find the following square roots. Round to two decimal places.

11. $\sqrt{13}$
12. $\sqrt{99}$
13. $\sqrt{123}$
14. $\sqrt{2}$
15. $\sqrt{2000}$
16. $\sqrt{.25}$
17. $\sqrt{1.35}$
18. $\sqrt{0.37}$
19. $\sqrt{0.7}$
20. $\sqrt{0.01}$