CHAPTER 3 Chapter 3: Graphing

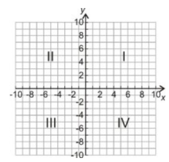
Chapter Outline

3.1 The Coordinate Plane

Learning Objectives

- Identify coordinates of points.
- Plot points in a coordinate plane.
- Graph a function given a table.
- Graph a function given a rule.

Introduction



We now make our transition from a number line that stretches in only one dimension (left to right) to something that exists in two dimensions. The **coordinate plane** can be thought of as two number lines that meet at right angles. The horizontal line is called the $x-$**axis** and the vertical line is the $y-$**axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**. The first quadrant (I) contains all the positive numbers from both axes. It is the top right quadrant. The other quadrants are numbered sequentially (II, III, IV) moving counterclockwise from the first.

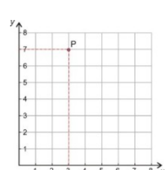
Identify Coordinates of Points

When given a point on a coordinate plane, it is a relatively easy task to determine its **coordinates**. The coordinates of a point are two numbers written together they are called an **ordered pair**. The numbers describe how far along the $x-$axis and $y-$axis the point is. The ordered pair is written in parenthesis, with the $x-$**coordinate** (also called the **ordinate**) first and the $y-$**coordinate** (or the **ordinate**) second.

$(1,7)$	An ordered pair with an x-value of one and a y-value of seven
$(0,5)$	an ordered pair with an x-value of zero and a y-value of five
$(-2.5,4)$	An ordered pair with an x-value of -2.5 and a y-value of four
$(-107.2,-.005)$	An ordered pair with an x-value of -107.2 and a y-value of -0.005.

The first thing to do is realize that identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example 1



Find the coordinates of the point labeled P in the diagram to the right.

Imagine you are standing at the origin (the points where the $x-$axis meets the $y-$axis). In order to move to a position where P was directly above you, you would move 3 units to the **right** (we say this is in the **positive** x direction).

The $x-$coordinate of P is $+3$.

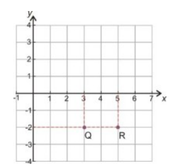
Now if you were standing at the three marker on the $x-$axis, point P would be 7 units above you (above the axis means it is in the **positive** y direction).

The $y-$coordinate of P is $+7$.

Solution

The coordinates of point P are $(3,7)$.

Example 2



Find the coordinates of the points labeled Q and R in the diagram to the right.

In order to get to Q we move three units to the right, in the positive$x$ direction, then two units **down**. This time we are moving in the **negative** y direction. The x coordinate of Q is $+3$, the y coordinate of Q is -2.

The coordinates of R are found in a similar way. The x coordinate is $+5$ (five units in positive x) and the $y-$coordinate is again -2.

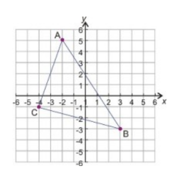
Solution

$Q(3,-2)$

$R(5,-2)$

Example 3

Triangle ABC is shown in the diagram to the right. Find the coordinates of the vertices A, B *and* C.



Point A:

$x-$coordinate $= -2$

$y-$coordinate $= +5$

Point B:

$x-$coordinate $= +3$

$y-$coordinate $= -3$

Point C:

$x-$coordinate $= -4$

$y-$coordinate $= -1$

Solution

$$A(-2,5)$$
$$B(3,-3)$$
$$C(-4,-1)$$

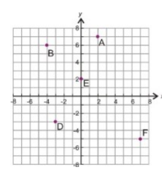
Plot Points in a Coordinate Plane

Plotting points is a simple matter once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example 4

Plot the following points on the coordinate plane.

$$A(2,7) \qquad B(-5,6) \qquad C(-6,0) \qquad D(-3,-3) \qquad E(0,2) \qquad F(7,-5)$$



Point $A(2,7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-5,6)$ is 5 units left, 6 units up. It is in Quadrant II.

Point $C(-6,0)$ is 6 units left, 0 units up. It is **on the** x **axis**.

Point $D(-3,-3)$ is 3 units left, 3 units down. It is in Quadrant III.

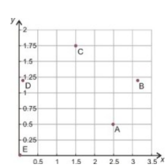
Point $E(0,2)$ is 2 units up from the origin. It is **on the** y **axis**.

Point $F(7,-5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Example 5

Plot the following points on the coordinate plane.

$$A(2.5,0.5) \qquad B(\pi,1.2) \qquad C(2,1.75) \qquad D(0.1,1.2) \qquad E(0,0)$$

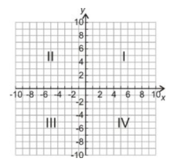


Choice of axes is always important. In Example Four, it was important to have all four quadrants visible. In this case, all the coordinates are positive. There is no need to show the negative values of x or y. Also, there are no x values bigger than about 3.14, and 1.75 is the largest value of y. We will therefore show these points on the following scale $0 \leq x \leq 3.5$ and $0 \leq y \leq 2$. The points are plotted to the right.

Here are some important points to note about this graph.

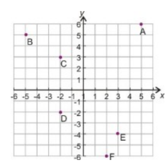
- The tick marks on the axes do not correspond to unit increments (i.e. the numbers do not go up by one).
- The scale on the $x-$axis is different than the scale on the $y-$axis.
- The scale is **chosen** to maximize the clarity of the plotted points.

Lesson Summary



- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the $x-$**axis**) and a vertical number line (the $y-$**axis**). The **origin** is the point where these two lines meet. Four areas, or **quadrants,** are formed as shown in the diagram at right.
- Each point on the coordinate plane has a set of **coordinates,** two numbers written as an **ordered pair** which describe how far along the $x-$axis and $y-$axis the point is. The $x-$**coordinate** is always written first, then the $y-$**coordinate.** Here is an exaxmple (x,y).

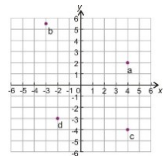
Review Questions



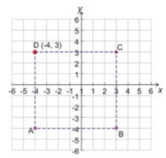
1. Identify the coordinates of each point, $A-F$, on the graph to the right.
2. Plot the following points on a graph and identify which quadrant each point lies in:
   (a) $(4,2)$
   (b) $(-3,5.5)$
   (c) $(4,-4)$
   (d) $(-2,-3)$
3. The following three points are three vertices of square $ABCD$. Plot them on a graph then determine what the coordinates of the fourth point, D, would be. Plot that point and label it.
   $A\ (-4,-4)$
   $B\ (3,-4)$
   $C\ (3,3)$
4. Becky has a large bag of MMs that she knows she should share with Jaeyun. Jaeyun has a packet of Starburst. Becky tells Jaeyun that for every Starburst he gives her, she will give him three MMs in return. If x is the number of Starburst that Jaeyun gives Becky, and $y$is the number of MMs he gets in return then complete each of the following.
   (a) Write an algebraic rule for y in terms of x
   (b) Make a table of values for y with x values of $0,1,2,3,4,5$.
   (c) Plot the function linking x *and* y on the following scale $0 \le x \le 10, 0 \le y \le 10$.

Review Answers

1. $A(5,6)B(-5,5)C(-2,3)D(-2,-2)E(3,-4)F(2,-6)$

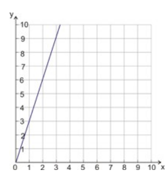
2. 

(a) Quadrant I
(b) Quadrant II
(c) Quadrant IV
(d) Quadrant III

3. 

4. (a) $y = 3x$
(b)

x	y
0	0
1	3
2	6
3	9
4	12
5	15

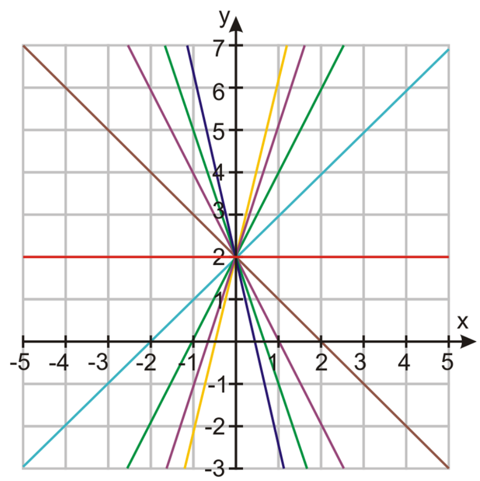
(c) 

3.2 Graphing Using Intercepts

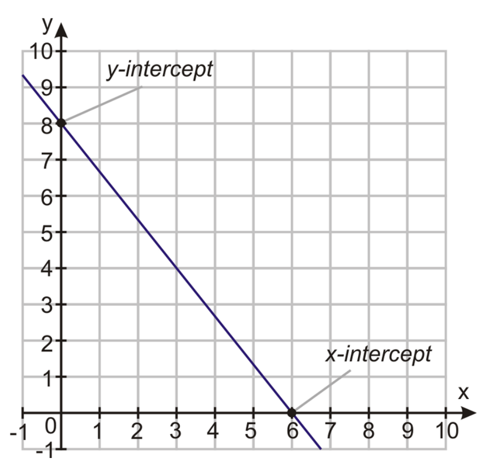
Learning Objectives

- Find intercepts of the graph of an equation.
- Use intercepts to graph an equation.
- Solve real-world problems using intercepts of a graph.

Introduction



Only two distinct points are needed to uniquely define a graph of a line. After all, there are an infinite number of lines that pass through a single point (a few are shown in the graph above). But if you supplied just one more point, there can only be one line that passes through both points. To plot the line, just plot the two points and use a ruler, edge placed on both points, to trace the graph of the line.

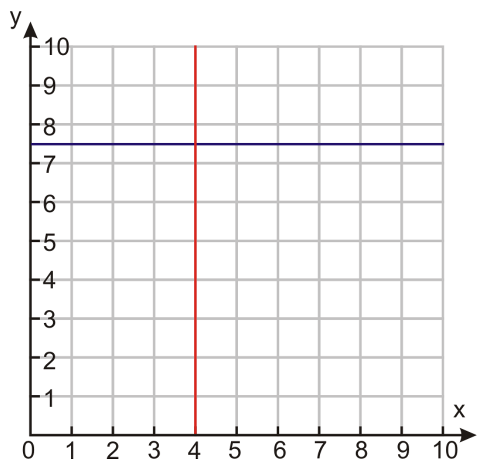


There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, we will focus on two points that are rather convenient for graphing: the points where our line crosses the x and y axes, or **intercepts.** We will be finding intercepts algebraically and using them to quickly plot graphs.

Look at the graph above. The $y-$**intercept** occurs at the point where the graph crosses the $y-$axis. The $y-$value at this point is 8.

Similarly the $x-$**intercept** occurs at the point where the graph crosses the $x-$axis. The $x-$value at this point is 6.

Now we know that the x value of all the points on the $y-$axis is zero, and the y value of all the points on the $x-$axis is also zero. So if we were given the coordinates of the two intercepts (0, 8) and (6, 0) we could quickly plot these points and join them with a line to recreate our graph.



Note: Not all lines will have both intercepts but most do. Specifically, horizontal lines never cross the $x-$axis and vertical lines never cross the $y-$axis. For examples of these special case lines, see the graph above.

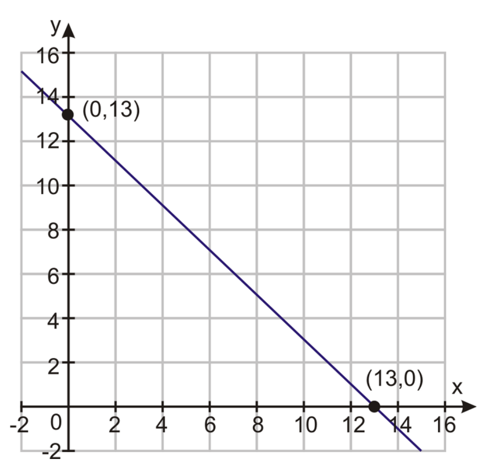
Finding Intercepts

Example 1

Find the intercepts of the line $y = 13 - x$ and use them to graph the function.

The first intercept is easy to find. The $y-$intercept occurs when $x = 0$ Substituting gives:

$$y = 13 - 0 = 13 \qquad (0, 13) \text{ is the } x - \text{intercept.}$$



We know that the $x-$intercept has, by definition, a $y-$value of zero. Finding the corresponding $x-$value is a simple case of substitution:

$$\begin{aligned} 0 &= 13 - x && \text{To isolate } x \text{ subtract 13 from both sides.} \\ -13 &= -x && \text{Divide by} - 1. \end{aligned}$$

Solution

(13, 0) is the $x-$intercept.

To draw the graph simply plot these points and join them with a line.

Example 2

Graph the following functions by finding intercepts.

a. $y = 2x + 3$

b. $y = 7 - 2x$

c. $4x - 2y = 8$

d. $2x + 3y = -6$

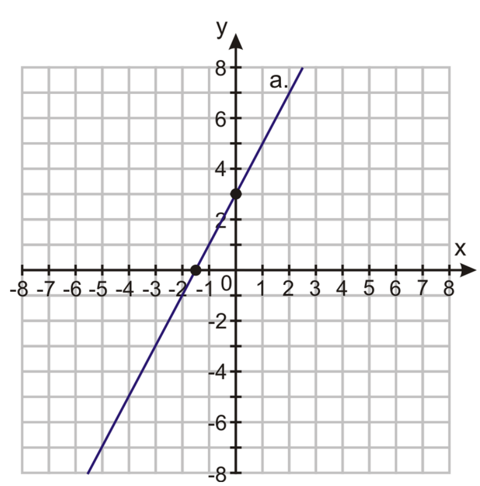
Solutions

a. Find the $y-$intercept by plugging in $x = 0$.

$$y = 2 \cdot 0 + 3 = 3 \qquad \text{The } y-\text{intercept is } (0, 3)$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 0 &= 2x + 3 && \text{Subtract 3 from both sides.} \\ -3 &= 2x && \text{Divide by 2.} \\ -\frac{3}{2} &= x && \text{The } x-\text{intercept is } (-1.5, 0). \end{aligned}$$

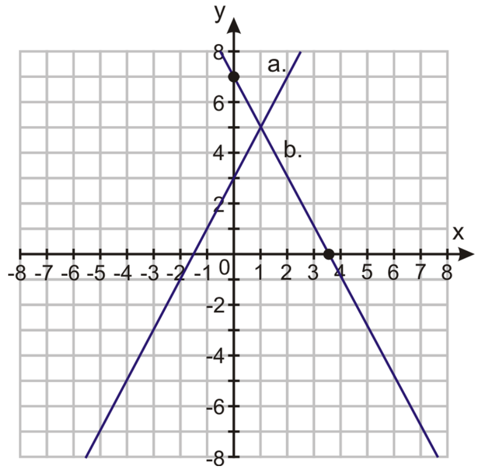


b. Find the $y-$intercept by plugging in $x = 0$.

$$y = 7 - 2 \cdot 0 = 7 \qquad \text{The } y-\text{intercept is } (0, 7).$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 0 &= 7 - 2x && \text{Subtract 7 from both sides.} \\ -7 &= -2x && \text{Divide by } -2. \\ \frac{7}{2} &= x && \text{The } x-\text{intercept is } (3.5, 0). \end{aligned}$$

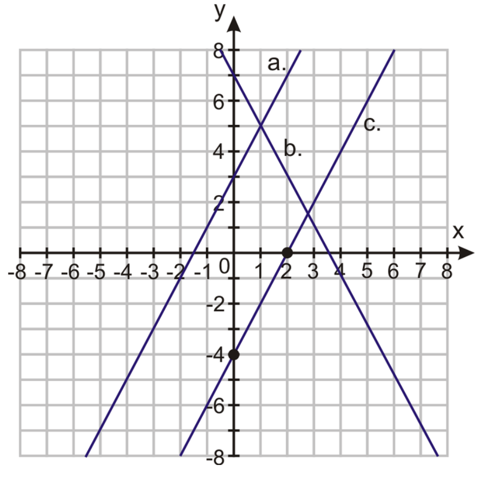


c. Find the $y-$intercept by plugging in $x = 0$.

$$\begin{aligned} 4 \cdot 0 - 2y &= 8 \\ -2y &= 8 && \text{Divide by } -2. \\ y &= -4 && \text{The } y-\text{intercept is } (0, -4). \end{aligned}$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 4x - 2 \cdot 0 &= 8 \\ 4x &= 8 && \text{Divide by 4.} \\ x &= 2 && \text{The } x-\text{intercept is } (2, 0). \end{aligned}$$

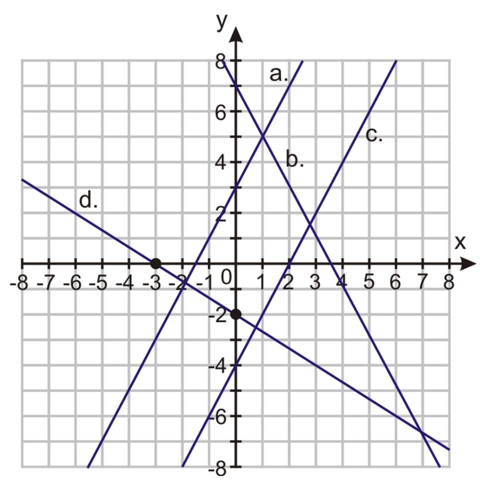


d. Find the $y-$intercept by plugging in $x = 0$.

$$\begin{aligned} 2 \cdot 0 + 3y &= -6 \\ 3y &= -6 && \text{Divide by 3.} \\ y &= -2 && \text{The } y-\text{intercept is } (0, -2). \end{aligned}$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 2x + 3 \cdot 0 &= -6 \\ 2x &= -6 && \text{Divide by 2.} \\ x &= -3 && \text{The } x-\text{intercept is } (-3, 0) \end{aligned}$$



Solving Real-World Problems Using Intercepts of a Graph

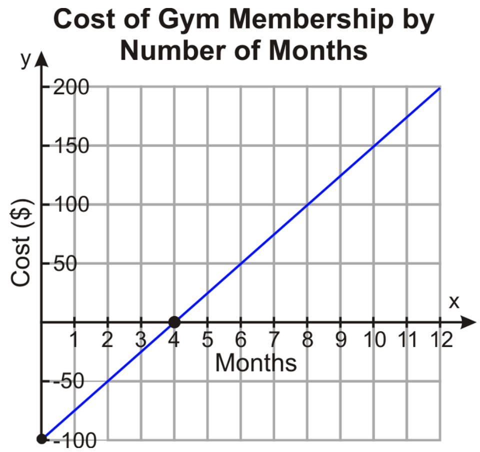
Example 3

The monthly membership cost of a gym is $25 per month. To attract members, the gym is offering a $100 cash rebate if members sign up for a full year. Plot the cost of gym membership over a 12 month period. Use the graph to determine the final cost for a 12 month membership.

Let us examine the problem. Clearly the cost is a function of the number of months (not the other way around). Our independent variable is the number of months (the domain will be whole numbers) and this will be our x value. The cost in dollars is the dependent variable and will be our y value. Every month that passes the money paid to the gym goes up by $25. However, we start with a $100 cash gift, so our **initial cost** ($y-$intercept) is $100. This pays for four months ($4 \times \$25 = 100$) so after four months the cost of membership ($y-$value) is zero.

The $y-$intercept is (0, -100). The $x-$intercept is (4, 0).

We plot our points, join them with a straight line and extend that line out all the way to the $x = 12$ line. The graph is shown below.



To find the cost of a 12 month membership we simply read off the value of the function at the 12 month point. A line drawn up from $x = 12$ on the x axis meets the function at a y value of \$200.

Solution

The cost of joining the gym for one year is \$200.

Example 4

Jesus has \$30 to spend on food for a class barbeque. Hot dogs cost \$0.75 each (including the bun) and burgers cost \$1.25 (including bun and salad). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than \$30.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that John buys is x, then the money spent on burgers is $1.25x$.

If the number of hot dogs he buys is y then the money spent on hot dogs is $0.75y$.

$$1.25x + 0.75y \qquad \text{The total cost of the food.}$$

The total amount of money he has to spend is \$30. If he is to spend it ALL, then we can use the following equation.

$$1.25x + 0.75y = 30$$

We solve for the intercepts using the cover-up method.

First the $y-$intercept $(x = 0)$.

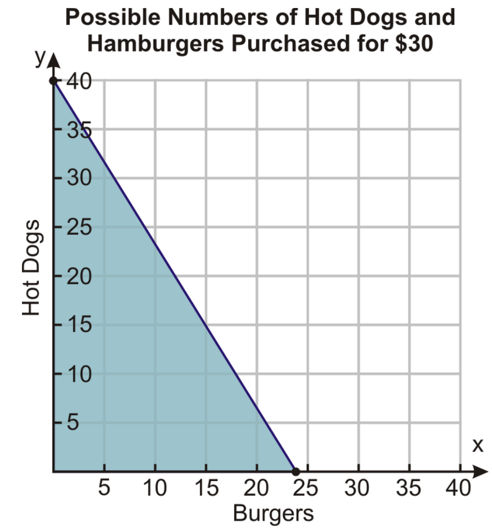
$1.25 \cdot 0 + 0.75y = 30$

$0.75y = 30$
$y = 40 \qquad y-\text{intercept } (0, 40)$

Then the $x-$intercept $(y = 0)$

$1.25x + 0.75 \cdot 0 = 30$

$1.25x = 30$
$x = 24 \qquad x-\text{intercept } (24, 0)$



We can now plot the points and join them to create our graph, shown right.

Here is an alternative to the equation method.

If Jesus were to spend ALL the money on hot dogs, he could buy $\frac{30}{0.75} = 40$ hot dogs. If on the other hand, he were to buy only burgers, he could buy $\frac{30}{1.25} = 24$ burgers. So you can see that we get two intercepts: (0 burgers, 40 hot dogs) and (24 burgers, 0 hot dogs). We would plot these in an identical manner and design our graph that way.

As a final note, we should realize that Jesus' problem is really an example of an **inequality.** He can, in fact, spend any amount up to $30. The only thing he cannot do is spend more than $30. So our graph reflects this. The shaded region shows where Jesus' solutions all lie. We will see more inequalities in a later section.

Lesson Summary

- A $y-$**intercept** occurs at the point where a graph crosses the $y-$axis $(x = 0)$ and an $x-$**intercept** occurs at the point where a graph crosses the $x-$axis $(y = 0)$.
- The $y-$intercept can be found by **substituting** $x = 0$ into the equation and solving for y. Likewise, the $x-$intercept can be found by **substituting** $y = 0$ into the equation and solving for x.
  - Note: A linear equation is in **standard form** if it is written as "positive coefficient times x plus (or minus) positive coefficient times y equals value". $(Ax + By = C)$

Review Questions

Find the intercepts for the following equations

1. $y = 3x - 6$
2. $y = -2x + 4$
3. $y = 14x - 21$
4. $y = 7 - 3x$
5. $5x - 6y = 15$
6. $3x - 4y = -5$
7. $2x + 7y = -11$
8. $5x + 10y = 25$

Find the intercepts and then graph the following equations.

9. $y = 2x + 3$

10. $6(x-1) = 2(y+3)$
11. $x-y=5$
12. $x+y=8$
13. At the local grocery store strawberries cost \$3.00 per pound and bananas cost \$1.00 per pound. If I have \$10 to spend between strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly \$10.
14. A movie theater charges \$7.50 for adult tickets and \$4.50 for children. If the theater takes \$900 in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
15. Why can't we use the intercept method to graph the following equation? $3(x+2) = 2(y+3)$

3.3 Slope

Here you'll learn how to find the slope of a line given the line's graph or two of its points.

What if you were given two points that a line passes through like (-1, 0) and (2, 2)? How could you find the slope of that line? After completing this Concept, you'll be able to find the slope of any line.

Guidance

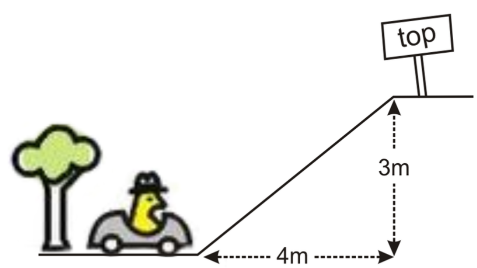
Wheelchair ramps at building entrances must have a slope between $\frac{1}{16}$ and $\frac{1}{20}$. If the entrance to a new office building is 28 inches off the ground, how long does the wheelchair ramp need to be?

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, or the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.

$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

To make it easier to remember, we often word it like this:

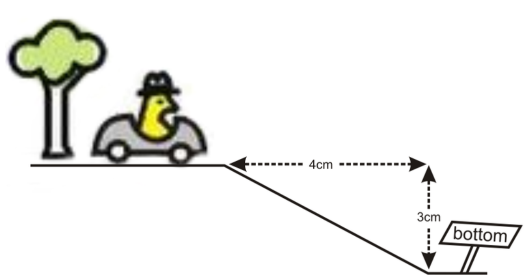
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$



In the picture above, the slope would be the ratio of the **height** of the hill to the horizontal **length** of the hill. In other words, it would be $\frac{3}{4}$, or 0.75.

If the car were driving to the **right** it would **climb** the hill - we say this is a positive slope. Any time you see the graph of a line that goes up as you move to the right, the slope is **positive**.

If the car kept driving after it reached the top of the hill, it might go down the other side. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**.



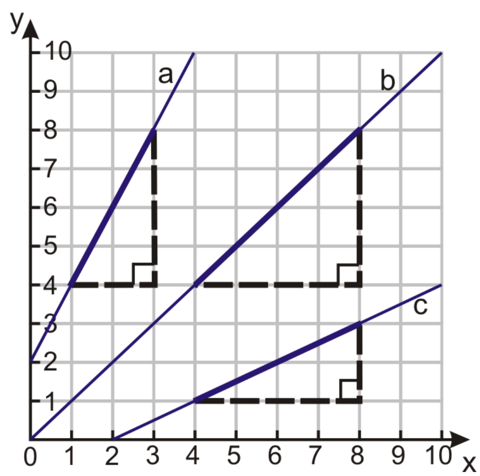
Here's where it gets tricky: If the car turned around instead and drove back down the left side of the hill, the slope of that side would still be positive. This is because the rise would be -3, but the run would be -4 (think of the $x-$axis - if you move from right to left you are moving in the **negative** $x-$direction). That means our slope ratio would be $\frac{-3}{-4}$, and the negatives cancel out to leave 0.75, the same slope as before. In other words, the slope of a line is the same no matter which direction you travel along it.

Find the Slope of a Line

A simple way to find a value for the slope of a line is to draw a right triangle whose hypotenuse runs along the line. Then we just need to measure the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example A

Find the slopes for the three graphs shown.



Solution

There are already right triangles drawn for each of the lines - in future problems you'll do this part yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. points whose coordinates are all integers).

a) The rise shown in this triangle is 4 units; the run is 2 units. The slope is $\frac{4}{2} = 2$.

b) The rise shown in this triangle is 4 units, and the run is also 4 units. The slope is $\frac{4}{4} = 1$.

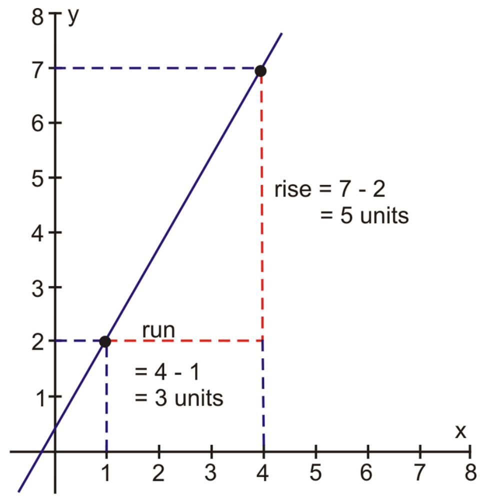
c) The rise shown in this triangle is 2 units, and the run is 4 units. The slope is $\frac{2}{4} = \frac{1}{2}$.

Example B

Find the slope of the line that passes through the points (1, 2) and (4, 7).

Solution

We already know how to graph a line if we're given two points: we simply plot the points and connect them with a line. Here's the graph:



Since we already have coordinates for the vertices of our right triangle, we can quickly work out that the rise is $7-2=5$ and the run is $4-1=3$ (see diagram). So the slope is $\frac{7-2}{4-1}=\frac{5}{3}$.

If you look again at the calculations for the slope, you'll notice that the 7 and 2 are the $y-$coordinates of the two points and the 4 and 1 are the $x-$coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1,y_1) and (x_2,y_2):

Slope between (x_1,y_1) and $(x_2,y_2) = \frac{y_2-y_1}{x_2-x_1}$

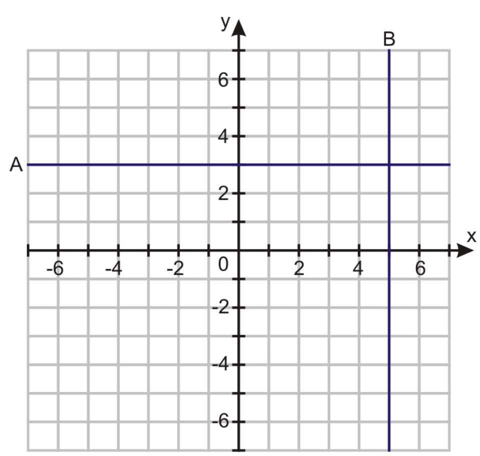
or $m=\frac{\Delta y}{\Delta x}$.

In the second equation the letter m denotes the slope (this is a mathematical convention you'll see often) and the Greek letter delta (Δ) means ***change***. So another way to express slope is *change in y* divided by *change in x*. In the next section, you'll see that it doesn't matter which point you choose as point 1 and which you choose as point 2.

Find the Slopes of Horizontal and Vertical lines

Example C

Determine the slopes of the two lines on the graph below.



Solution

There are 2 lines on the graph: $A(y=3)$ and $B(x=5)$.

Let's pick 2 points on line A—say, $(x_1,y_1)=(-4,3)$ and $(x_2,y_2)=(5,3)$—and use our equation for slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (3)}{(5) - (-4)} = \frac{0}{9} = 0.$$

If you think about it, this makes sense - if y doesn't change as x increases then there is no slope, or rather, the slope is zero. You can see that this must be true for all horizontal lines.

Horizontal lines (y = *constant*) all have a slope of 0.

Now let's consider line B. If we pick the points $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (5, 4)$, our slope equation is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4)-(-3)}{(5)-(5)} = \frac{7}{0}$. But dividing by zero isn't allowed!

In math we often say that a term which involves division by zero is **undefined.** (Technically, the answer can also be said to be infinitely large—or infinitely small, depending on the problem.)

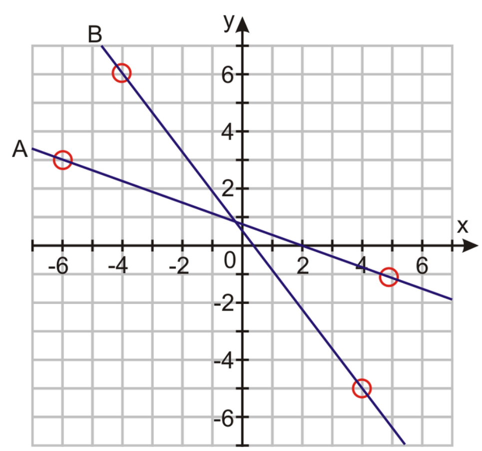
Vertical lines ($x = constant$) all have an infinite (or undefined) slope.

Vocabulary

- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as "m".
- Slope can be expressed as $\frac{\text{rise}}{\text{run}}$, or $\frac{\Delta y}{\Delta x}$.
- The slope between two points (x_1, y_1) and (x_2, y_2) is equal to $\frac{y_2 - y_1}{x_2 - x_1}$.
- **Horizontal lines** (where $y = a$ constant) all have a slope of 0.
- **Vertical lines** (where $x = a$ constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity.**

Guided Practice

Find the slopes of the lines on the graph below.



Solution

Look at the lines - they both slant down (or decrease) as we move from left to right. Both these lines have **negative slope.**

The lines don't pass through very many convenient lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been circled on the graph, and we'll use them to determine the slope. We'll also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For Line A:

$$(x_1,y_1)=(-6,3) \qquad (x_2,y_2)=(5,-1)$$
$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(-1)-(3)}{(5)-(-6)}=\frac{-4}{11}\approx -0.364$$

$$(x_1,y_1)=(5,-1) \qquad (x_2,y_2)=(-6,3)$$
$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(3)-(-1)}{(-6)-(5)}=\frac{4}{-11}\approx -0.364$$

For Line B

$$(x_1,y_1)=(-4,6) \qquad (x_2,y_2)=(4,-5)$$
$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(-5)-(6)}{(4)-(-4)}=\frac{-11}{8}=-1.375$$

$$(x_1,y_1)=(4,-5) \qquad (x_2,y_2)=(-4,6)$$
$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(6)-(-5)}{(-4)-(4)}=\frac{11}{-8}=-1.375$$

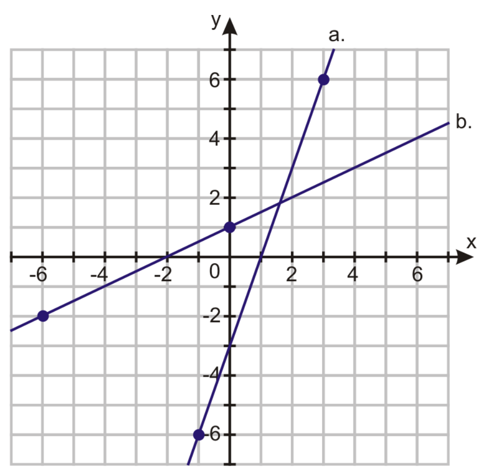
You can see that whichever way round you pick the points, the answers are the same. Either way, **Line *A* has slope -0.364, and Line *B* has slope -1.375.**

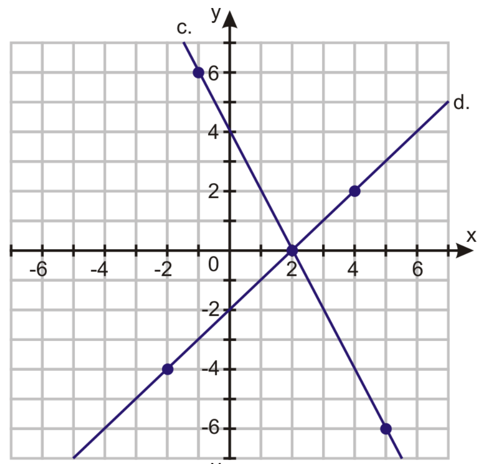
Practice

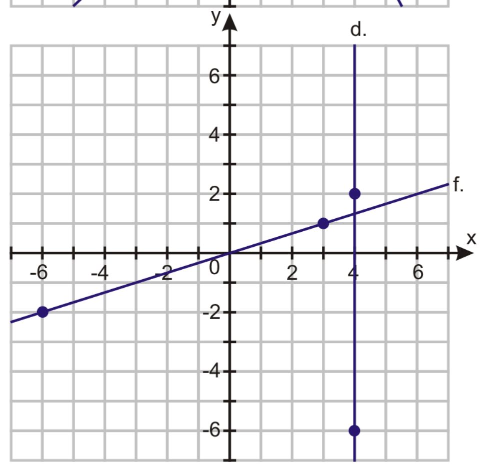
Use the slope formula to find the slope of the line that passes through each pair of points.

1. (-5, 7) and (0, 0)
2. (-3, -5) and (3, 11)
3. (3, -5) and (-2, 9)
4. (-5, 7) and (-5, 11)
5. (9, 9) and (-9, -9)
6. (3, 5) and (-2, 7)
7. (2.5, 3) and (8, 3.5)

For each line in the graphs below, use the points indicated to determine the slope.

8. 

9. 

10. 

11. For each line in the graphs above, imagine another line with the same slope that passes through the point (1, 1), and name one more point on that line.

3.4 Using Slope-Intercept Form

Here you'll use slope-intercept form and identify the slope and the y-intercept.

Guidance

We have seen linear equations in function form, have created tables of values and graphs to represent them, looked at their x- and y-intercepts, and studied their slopes. One of the most useful forms of a linear equation is the ***slope-intercept form*** which we will be using with standard form in this Concept.

Remember ***standard form***?

The standard form of an equation is when the equation is written in $Ax+By=C$ form.

This form of the equation allows us to find many possible solutions. In essence, we could substitute any number of values for x and y and create the value for C. When an equation is written in standard form, it is challenging for us to determine the slope and the y –intercept.

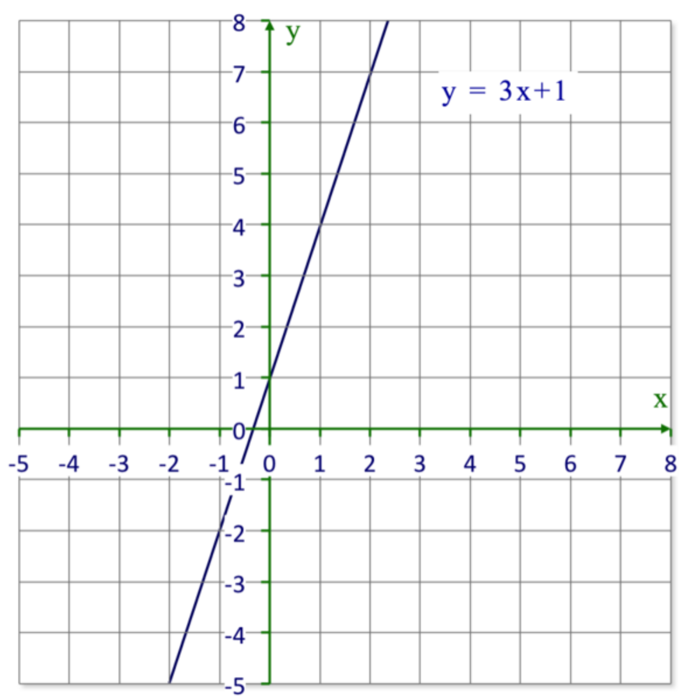
Think back, remember that the *slope* is the steepness of the line and the y *–intercept* is the point where the line crosses the y –axis.

We can write an equation in a different form than in standard form. This is when $y =$ an equation. We call this form of an equation *slope –intercept form* .

Slope –Intercept Form is $y = mx+b$ –where m is the slope and b is the y –intercept.

Take a look at this graph and equation.

Graph the line $y = 3x+1$



Here we can calculate the slope of the line using the rise over the run and see that it is 3. The y –intercept is 1. Notice that we can find these values in our equation too.

When an equation is in slope –intercept form, we can spot the slope and the y –intercept by looking at the equation.

$y = mx + b$

Here m is the value of the slope and b is the value of the y –intercept.

For any equation written in the form $y = mx + b$, m is the slope and b is the y-intercept. For that reason, $y = mx + b$ is called the *slope-intercept form*. Using the properties of equations, you can write any equation in this form.

Because we can use slope –intercept form, we can rewrite equations in standard form into slope –intercept form. Then we can easily determine the slope and y –intercept of each equation.

Take a look here.

Write $4x + 2y = 6$ in slope –intercept form. Then determine the slope and the y –intercept by using the equation.

$$\begin{aligned}
4x + 2y &= 6 \\
4x + 2y - 2y &= 6 - 2y \\
4x &= 6 - 2y \\
4x - 6 &= -2y \\
\frac{4x - 6}{-2} &= y \\
y &= -2x + 3
\end{aligned}$$

Now we can determine the slope and the y –intercept from the equation.

$$\begin{aligned}
-2 &= \text{slope} \\
3 &= y - \text{intercept}
\end{aligned}$$

Think back to our work with functions. Remember how we could write a function in function form? Well take a look at function form compared with slope –intercept form.

Function form $= f(x) = 2x + 1$

Slope –Intercept Form $= y = 2x + 1$

Yes! The two are the same. These two equations are equivalent!

Determine the slope and the y-intercept in each equation.

Example A

$y = x + 4$

Solution: slope = 1, y-intercept = 4

Example B

$2x + y = 10$

Solution: slope = -2, y-intercept = 10

Example C

$-3x+y=9$

Solution: slope = 3, y-intercept = 9

Now let's go back to the dilemma at the beginning of the Concept.

$y=-2x-8$

Looking at this equation, you can see that the slope is -2 and the y-intercept is 8.

Vocabulary

Slope –Intercept Form
the form of an equation $y=mx+b$

Standard Form
the form of an equation $Ax+By=C$

Slope
the steepness of the line, calculated by the ratio of rise over run.

y –Intercept
the point where a line crosses the $y$axis.

Guided Practice

Here is one for you to try on your own.

Write this equation in slope-intercept form and then determine the slope and the y-intercept.

$$\begin{aligned}
-18x+6y &= 12 \\
-18x+6y &= 12 \\
-18x+6y+18x &= 18x+12 \\
6y &= 18x+12 \\
6y &= 18x+12 \\
\frac{18x+12}{6} &= y \\
y &= 3x+2
\end{aligned}$$

Given this equation, the slope is 3 and the y-intercept is 2.

Practice

Directions: Look at each equation and identify the slope and the y –intercept by looking at each equation. There are two answers for each problem.

1. $y = 2x + 4$
2. $y = 3x - 2$
3. $y = 4x + 3$
4. $y = 5x - 1$
5. $y = \frac{1}{2}x + 2$
6. $y = -2x + 4$
7. $y = -3x - 1$
8. $y = \frac{-1}{3}x + 5$

Directions: Use what you have learned to write each in slope –intercept form and then answer each question.

9. $2x + 4y = 12$
10. Write this equation in slope –intercept form.
11. What is the slope?
12. What is the y –intercept?
13. $6x + 3y = 24$
14. Write this equation in slope –intercept form.
15. What is the slope?
16. What is the y –intercept?
17. $5x + 5y = 15$
18. Write this equation in slope –intercept form.
19. What is the slope?
20. What is the y –intercept?

3.5 Direct Variation Models

Learning Objectives

- Identify direct variation.
- Graph direct variation equations.
- Solve real-world problems using direct variation models.

Introduction

Suppose you see someone buy five pounds of strawberries at the grocery store. The clerk weighs the strawberries and charges \$12.50 for them. Now suppose you wanted two pounds of strawberries for yourself. How much would you expect to pay for them?

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a "per pound" basis, and that if you buy two-fifths of the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries.

$$\frac{2}{5} \times \$12.50 = \$5.00$$

Similarly, if you bought 10 pounds of strawberries (twice the amount) you would pay $2 \times \$12.50$ and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x, then we write the relationship as:

$$y = k \cdot x$$

k is called the **constant of proportionality**.

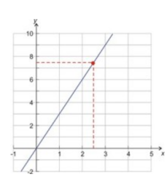
If we were to graph this function you can see that it passes through the origin, because $y = 0$, when $x = 0$ whatever the value of k. So we know that a direct variation, when graphed, has a single intercept at $(0,0)$.

Example 1

If y varies directly with x according to the relationship $y = k \cdot x$, and $y = 7.5$ when $x = 2.5$, determine the constant of proportionality, k.

We can solve for the constant of proportionality using substitution.

Substitute $x = 2.5$ and $y = 7.5$ into the equation $y = k \cdot x$



www.ck12.org/saythanks

$$
\begin{aligned}
7.5 &= k(2.5) && \text{Divide both sides by 2.5.} \\
\frac{7.5}{2.5} &= k = 3
\end{aligned}
$$

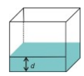
Solution

The constant of proportionality, $k = 3$.

We can graph the relationship quickly, using the intercept $(0,0)$ and the point $2.5,7.5)$. The graph is shown right. It is a straight line with a slope $= 3$.

The graph of a direct variation has a slope that is equal to the constant of proportionality, k.

Example 2



The volume of water in a fish-tank, V, varies directly with depth, d. If there are 15 *gallons in the tank when the depth is eight inches, calculate how much water is in the tank when the depth is* 20 *inches.*

This is a good example of a direct variation, but for this problem we will need to determine the equation of the variation ourselves. Since the volume, V, depends on depth, d, we will use the previous equation to create new one that is better suited to the content of the new problem.

$$
\begin{aligned}
y &= k \cdot x && \text{In place of } y \text{ we will use } V \text{ and in place of } x \text{ we will use d.} \\
V &= k \cdot d
\end{aligned}
$$

We know that when the depth is 8 inches, the volume is 15 gallons. Now we can substitute those values into our equation.

Substitute $V = 15$ and $x = 8$:

$$
\begin{aligned}
V &= k \cdot d \\
15 &= k(8) && \text{Divide both sides by8.} \\
\frac{15}{8} &= k = 1.875
\end{aligned}
$$

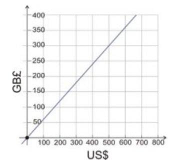
Now to find the volume of water at the final depth we use $V = k \cdot d$ and substitute for our new d.

$$
\begin{aligned}
V &= k \cdot d \\
V &= 1.875 \times 20 \\
V &= 37.5
\end{aligned}
$$

Solution

At a depth of 20 inches, the volume of water in the tank is 37.5 gallons.

Example 3



The graph shown to the right shows a conversion chart used to convert U.S. dollars (US\$) to British pounds (GB) in a bank on a particular day. Use the chart to determine the following.

(i) The number of pounds you could buy for \$600.

(ii) The number of dollars it would cost to buy 200.

(iii) The exchange rate in pounds per dollar.

(iv) Is the function continuous or discrete?

Solution

In order to solve (i) and (ii) we could simply read off the graph: it looks as if at $x = 600$ the graph is about one fifth of the way between 350 and 400. So \$600 would buy 360. Similarly, the line $y = 200$ would appear to intersect the graph about a third of the way between \$300 and \$400. We would probably round this to \$330. So it would cost approximately \$330 to buy 200.

To solve for the exchange rate we should note that as this is a direct variation, because the graph is a straight line passing through the origin. The slope of the line gives the constant of proportionality (in this case the **exchange rate**) and it is equal to the ratio of the $y-$value to $x-$value. Looking closely at the graph, it is clear that there is one lattice point that the line passes through $(500, 300)$. This will give us the most accurate estimate for the slope (exchange rate).

$$y = k \cdot x \Rightarrow k = \frac{y}{x}$$

$$\text{rate} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar}$$

Graph Direct Variation Equations

We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of proportionality, k. Graphing is a simple matter of using the point-slope or point-point methods discussed earlier in this chapter.

Example 4

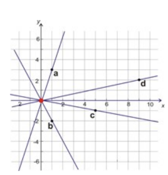
Plot the following direct relations on the same graph.

a. $y = 3x$

b. $y = -2x$

c. $y = -0.2x$

d. $y = \frac{2}{9}x$



Solution

a. The line passes through $(0,0)$. All these functions will pass through this point. It is plotted in red. This function has a slope of 3. When we move across by one unit, the function increases by three units.

b. The line has a slope of -2. When we move across the graph by one unit the function **falls** by two units.

c. The line has a slope of -0.2. As a fraction this is equal to $-\frac{1}{5}$ When we move across by five units, the function **falls** by one unit.

d. The line passes through $(0,0)$ and has a slope of $\frac{2}{9}$. When we move across the graph by 9 units, the function increases by two units.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time that we have one quantity that doubles when another related quantity doubles, we say that they follow a direct variation.

Newton's Second Law

In 1687, Sir Isaac Newton published the famous *Principea Mathematica*. It contained, among other things, his Second Law of Motion. This law is often written as:

$$F = m \cdot a$$

A force of F (Newtons) applied to a mass of m (kilograms) results in acceleration of $a(\text{meterspersecond}^2)$.

Example 5

If a 175 *Newton force causes a heavily loaded shopping cart to accelerate down the aisle with an acceleration of* $2.5\ \text{m/s}^2$, *calculate*

(i) The mass of the shopping cart.

(ii) The force needed to accelerate the same cart at $6\ \text{m/s}^2$.

Solution

(i) This question is basically asking us to solve for the constant of proportionality. Let us compare the two formulas.

$$\begin{aligned} y &= k \cdot x && \text{The direct variation equation} \\ F &= m \cdot a && \text{Newton's Second law} \end{aligned}$$

We see that the two equations have the same form; y is analogous to force and x analogous to acceleration.

We can solve for m (the mass) by substituting our given values for force and acceleration:

Substitute $F = 175$, $a = 2.5$

$$\begin{aligned} 175 &= m(2.5) && \text{Divide both sides by 2.5.} \\ 70 &= m \end{aligned}$$

The mass of the shopping cart is 70 kg.

(ii) Once we have solved for the mass we simply substitute that value, plus our required acceleration back into the formula $F = m \cdot a$ and solve for F:

Substitute $m = 70$, $a = 6$

$$F = 70 \times 6 = 420$$

The force needed to accelerate the cart at 6 m/s^2 is 420 Newtons.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship:

$$V = I \cdot R \qquad \text{where R is the resistance (measured in Ohms)}$$

The resistance is considered to be a constant for all values of V and I.

Example 6

A certain electronics component was found to pass a current of 1.3 *amps at a voltage of* 2.6 *volts. When the voltage was increased to* 12.0 *volts the current was found to be* 6.0 *amps.*

a) Does the component obey Ohms law?

b) What would the current be at 6 *volts?*

Solution

a) Ohm's law is a simple direct proportionality law. Since the resistance R is constant, it acts as our constant of proportionality. In order to know if the component obeys Ohm's law we need to know if it follows a direct proportionality rule. In other words **is** V **directly proportional to** I?

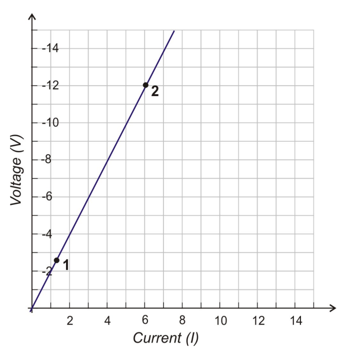


FIGURE 3.1

Method One Graph It

If we plot our two points on a graph and join them with a line, does the line pass through $(0,0)$?

Point 1 $= 2.6$, $I = 1.3$ our point is $(1.3, 2.6)^*$

Point 2 $V = 12.0$, $I = 6.0$ our point is $(6, 12)$

Plotting the points and joining them gives the following graph.

The graph does appear to pass through the origin, so

Yes, the component obeys Ohms law.

Method Two Solve for

We can quickly determine the value of R in each case. It is the ratio of the voltage to the resistance.

Case 1 $R = \frac{V}{I} = \frac{2.6}{1.3} = 2$ Ohms

Case 2 $R = \frac{V}{I} = \frac{12}{6} = 2$ Ohms

The values for R agree! This means that the line that joins point 1 to the origin is the same as the line that joins point 2 to the origin. **The component obeys Ohms law**.

b) To find the current at 6 volts, simply substitute the values for V and R into $V = I \cdot R$

Substitute $V = 6$, $R = 2$

- In physics, it is customary to plot voltage on the horizontal axis as this is most often the independent variable. In that situation, the slope gives the **conductance,** σ. However, by plotting the current on the horizontal axis, the **slope** is equal to the **resistance,** R.

$$6 = I(2) \qquad \text{Divide both sides by 2.}$$
$$3 = I$$

Solution

The current through the component at a voltage of 6 volts is 3 amps.

Lesson Summary

- If a variable y varies ***directly*** with variable x, then we write the relationship as

$$y = k \cdot x$$

Where k is a constant called the **constant of proportionality.**

- **Direct variation** is very common in many areas of science.

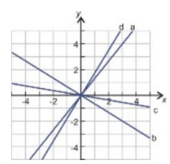
Review Questions

1. Plot the following direct variations on the same graph.
   (a) $y = \frac{4}{3}x$
   (b) $y = -\frac{2}{3}x$
   (c) $y = -\frac{1}{6}x$

(d) $y = 1.75x$

2. Dasans mom takes him to the video arcade for his birthday. In the first 10 minutes, he spends \$3.50 playing games. If his allowance for the day is \$20.00, how long can he keep playing games before his money is gone?
3. The current standard for low-flow showerheads heads is 2.5 gallons per minute. Calculate how long it would take to fill a 30 gallon bathtub using such a showerhead to supply the water.
4. Amen is using a hose to fill his new swimming pool for the first time. He starts the hose at 10 P.M. and leaves it running all night. At 6 AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
5. Land in Wisconsin is for sale to property investors. A 232 acre lot is listed for sale for \$200500. Assuming the same price per acre, how much would a 60 acre lot sell for?
6. The force (F) needed to stretch a spring by a distance x is given by the equation $F = k \cdot x$, where k is the spring constant (measured in Newtons per centimeter, N/cm). If a 12 Newton force stretches a certain spring by 10 cm, calculate:
   (a) The spring constant, k
   (b) The force needed to stretch the spring by 7 cm .
   (c) The distance the spring would stretch with a 23 Newton force.

Review Answers

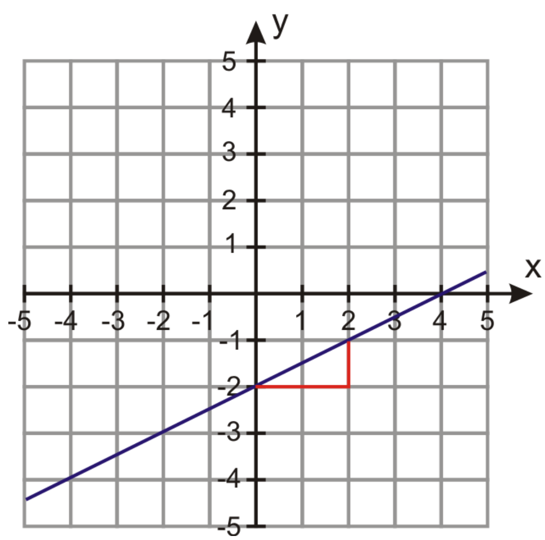
1. 

2. 57 minutes 8 seconds
3. 12 minutes
4. 12 : 00 Midday
5. \$51, 853
6. (a) $k = 1.2$ N/cm
   (b) 8.4 Newtons
   (c) 19.17 cm

3.6 Forms of Linear Equations

You've already learned how to write an equation in slope–intercept form: simply start with the general equation for the slope-intercept form of a line, $y = mx + b$, and then plug the given values of $m$and $b$into the equation. For example, a line with a slope of 4 and a $y-$intercept of -3 would have the equation $y = 4x - 3$.

If you are given just the graph of a line, you can read off the slope and $y-$intercept from the graph and write the equation from there. For example, on the graph below you can see that the line rises by 1 unit as it moves 2 units to the right, so its slope is $\frac{1}{2}$. Also, you can see that the $y-$intercept is -2, so the equation of the line is $y = \frac{1}{2}x - 2$.



Write an Equation Given the Slope and a Point

Often, we don't know the value of the $y-$intercept, but we know the value of y for a non-zero value of x. In this case, it's often easier to write an equation of the line in **point-slope form.** An equation in point-slope form is written as $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the line.

Example 1

A line has a slope of $\frac{3}{5}$, and the point (2, 6) is on the line. Write the equation of the line in point-slope form.

Solution

Start with the formula $y - y_0 = m(x - x_0)$.

Plug in $\frac{3}{5}$ for m, 2 for x_0 and 6 for y_0.

The equation in point-slope form is $y - 6 = \frac{3}{5}(x - 2)$.

Notice that the equation in point-slope form is not solved for y. If we did solve it for y, we'd have it in $y-$intercept form. To do that, we would just need to distribute the $\frac{3}{5}$ and add 6 to both sides. That means that the equation of this line in slope-intercept form is $y = \frac{3}{5}x - \frac{6}{5} + 6$, or simply $y = \frac{3}{5}x + \frac{24}{5}$.

Write an Equation Given Two Points

Starting with the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, we plug in the $x-$and $y-$values of the two points to get $m = \frac{2-3}{5-(-2)} = \frac{-1}{7}$. We can plug that value of $m$into the point-slope formula to get $y - y_0 = -\frac{1}{7}(x - x_0)$.*For example, suppose we are told that the line passes through the points (-2, 3) and (5, 2). To find the equation of the line, we can start by finding the slope.*

Now we just need to pick one of the two points to plug into the formula. Let's use (5, 2); that gives us $y - 2 = -\frac{1}{7}(x-5)$.

What if we'd picked the other point instead? Then we'd have ended up with the equation $y - 3 = -\frac{1}{7}(x+2)$, which doesn't look the same. That's because there's more than one way to write an equation for a given line in point-slope form. But let's see what happens if we solve each of those equations for y.

Starting with $y - 2 = -\frac{1}{7}(x-5)$, we distribute the $-\frac{1}{7}$ and add 2 to both sides. That gives us $y = -\frac{1}{7}x + \frac{5}{7} + 2$, or $y = -\frac{1}{7}x + \frac{19}{7}$.

On the other hand, if we start with $y - 3 = -\frac{1}{7}(x+2)$, we need to distribute the $-\frac{1}{7}$ and add 3 to both sides. That gives us $y = -\frac{1}{7}x - \frac{2}{7} + 3$, which also simplifies to $y = -\frac{1}{7}x + \frac{19}{7}$.

So whichever point we choose to get an equation in point-slope form, the equation is still mathematically the same, and we can see this when we convert it to $y-$intercept form.

Example 2

A line contains the points (3, 2) and (-2, 4). Write an equation for the line in point-slope form; then write an equation in y−intercept form.

Solution

Find the slope of the line: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{-2-3} = -\frac{2}{5}$

Plug in the value of the slope: $y - y_0 = -\frac{2}{5}(x - x_0)$.

Plug point (3, 2) into the equation: $y - 2 = -\frac{2}{5}(x-3)$.

The equation in point-slope form is $y - 2 = -\frac{2}{5}(x-3)$.

To convert to $y-$intercept form, simply solve for y:

$$y - 2 = -\frac{2}{5}(x-3) \rightarrow y - 2 = -\frac{2}{5}x - \frac{6}{5} \rightarrow y = -\frac{2}{5}x - \frac{6}{5} + 2 = -\frac{2}{5}x + \frac{4}{5}.$$

The equation in $y-$intercept form is $y = -\frac{2}{5}x + \frac{4}{5}$.

Graph an Equation in Point-Slope Form

Another useful thing about point-slope form is that you can use it to graph an equation without having to convert it to slope-intercept form. From the equation $y - y_0 = m(x - x_0)$, you can just read off the slope m and the point (x_0, y_0). To draw the graph, all you have to do is plot the point, and then use the slope to figure out how many units up and over you should move to find another point on the line.

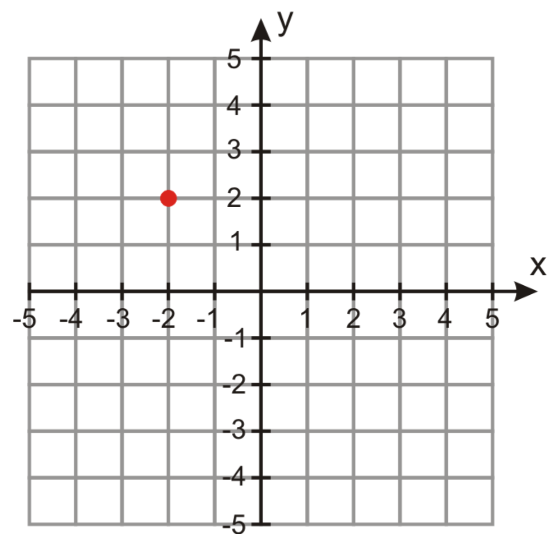
Example 5

Make a graph of the line given by the equation $y + 2 = \frac{2}{3}(x-2)$.

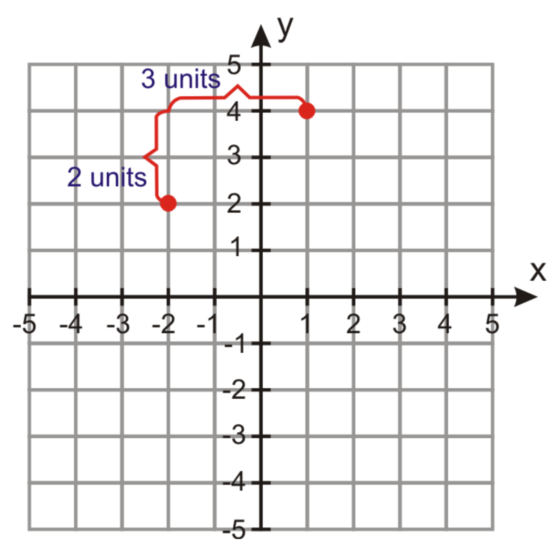
Solution

To read off the right values, we need to rewrite the equation slightly: $y-(-2)=\frac{2}{3}(x-2)$. Now we see that point (2, -2) is on the line and that the slope is $\frac{2}{3}$.

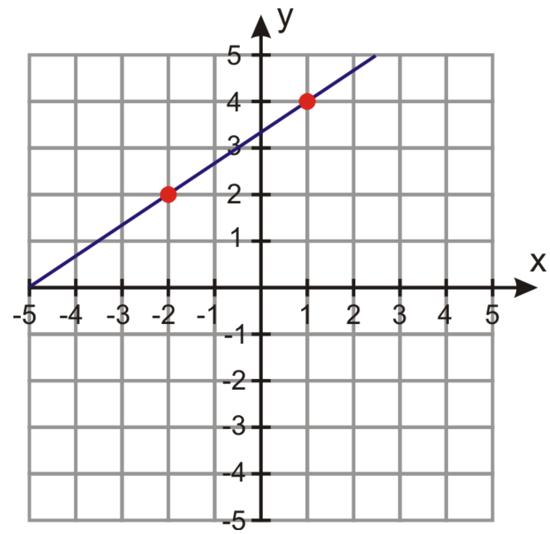
First plot point (2, -2) on the graph:



A slope of $\frac{2}{3}$ tells you that from that point you should move 2 units up and 3 units to the right and draw another point:



Now draw a line through the two points and extend it in both directions:



One useful thing about standard form is that it allows us to write equations for vertical lines, which we can't do in slope-intercept form.You've already encountered another useful form for writing linear equations: **standard form.** An equation in standard form is written $ax+by=c$, where a, b, and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

For example, let's look at the line that passes through points (2, 6) and (2, 9). How would we find an equation for that line in slope-intercept form?

First we'd need to find the slope: $m=\frac{9-6}{0-0}=\frac{3}{0}$. But that slope is undefined because we can't divide by zero. And if we can't find the slope, we can't use point-slope form either.

If we just graph the line, we can see that x equals 2 no matter what y is. There's no way to express that in slope-intercept or point-slope form, but in standard form we can just say that $x+0y=2$, or simply $x=2$.

Converting to Standard Form

To convert an equation from another form to standard form, all you need to do is rewrite the equation so that all the variables are on one side of the equation and the coefficient of x is not negative.

Example 1

Rewrite the following equations in standard form:

a) $y=5x-7$

b) $y-2=-3(x+3)$

c) $y=\frac{2}{3}x+\frac{1}{2}$

Solution

We need to rewrite each equation so that all the variables are on one side and the coefficient of x is not negative.

a) $y=5x-7$

Subtract y from both sides to get $0=5x-y-7$.

Add 7 to both sides to get $7=5x-y$.

Flip the equation around to put it in standard form: $5x-y=7$.

b) $y-2=-3(x+3)$

Distribute the –3 on the right-hand-side to get $y-2=-3x-9$.

Add $3x$ to both sides to get $y+3x-2=-9$.

Add 2 to both sides to get $y+3x=-7$. Flip that around to get $3x+y=-7$.

c) $y=\frac{2}{3}x+\frac{1}{2}$

Find the common denominator for all terms in the equation –in this case that would be 6.

Multiply all terms in the equation by 6: $6\left(y=\frac{2}{3}x+\frac{1}{2}\right) \Rightarrow 6y=4x+3$

Subtract $6y$ from both sides: $0=4x-6y+3$

Subtract 3 from both sides: $-3=4x-6y$

The equation in standard form is $4x-6y=-3$.

Graphing Equations in Standard Form

When an equation is in slope-intercept form or point-slope form, you can tell right away what the slope is. How do you find the slope when an equation is in standard form?

Well, you could rewrite the equation in slope-intercept form and read off the slope. But there's an even easier way. Let's look at what happens when we rewrite an equation in standard form.

Starting with the equation $ax+by=c$, we would subtract ax from both sides to get $by=-ax+c$. Then we would divide all terms by b and end up with $y=-\frac{a}{b}x+\frac{c}{b}$.

That means that the slope is $-\frac{a}{b}$ and the $y-$intercept is $\frac{c}{b}$. So next time we look at an equation in standard form, we don't have to rewrite it to find the slope; we know the slope is just $-\frac{a}{b}$, where a and b are the coefficients of x and y in the equation.

Example 2

Find the slope and the y−intercept of the following equations written in standard form.

a) $3x+5y=6$

b) $2x-3y=-8$

c) $x-5y=10$

Solution

a) $a=3, b=5$, and $c=6$, so the slope is $-\frac{a}{b}=-\frac{3}{5}$, and the $y-$intercept is $\frac{c}{b}=\frac{6}{5}$.

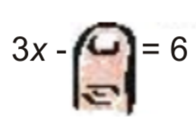
b) $a=2, b=-3$, and $c=-8$, so the slope is $-\frac{a}{b}=\frac{2}{3}$, and the $y-$intercept is $\frac{c}{b}=\frac{8}{3}$.

c) $a=1, b=-5$, and $c=10$, so the slope is $-\frac{a}{b}=\frac{1}{5}$, and the $y-$intercept is $\frac{c}{b}=\frac{10}{-5}=-2$.

Once we've found the slope and $y-$intercept of an equation in standard form, we can graph it easily. But if we start with a graph, how do we find an equation of that line in standard form?

First, remember that we can also use the cover-up method to graph an equation in standard form, by finding the intercepts of the line. For example, let's graph the line given by the equation $3x-2y=6$.

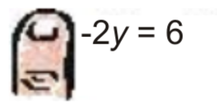
To find the $x-$intercept, cover up the y term (remember, the $x-$intercept is where $y=0$):



$3x=6 \Rightarrow x=2$

The $x-$ intercept is (2, 0).

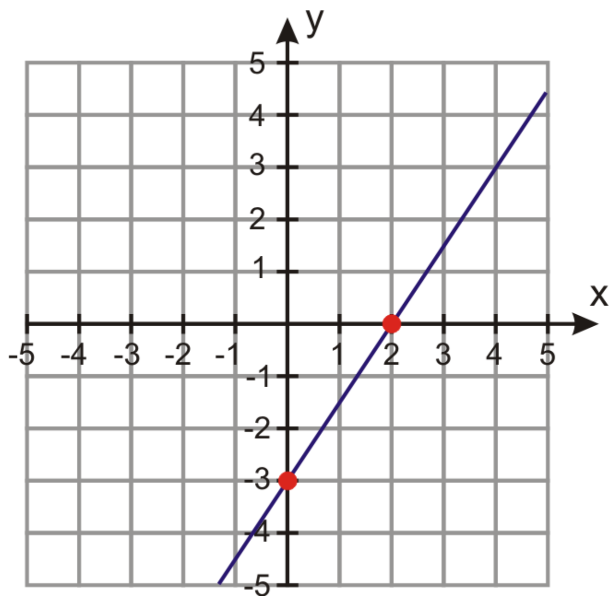
To find the $y-$intercept, cover up the x term (remember, the $y-$intercept is where $x = 0$):



$-2y = 6 \Rightarrow y = -3$

The $y-$intercept is (0, -3).

We plot the intercepts and draw a line through them that extends in both directions:

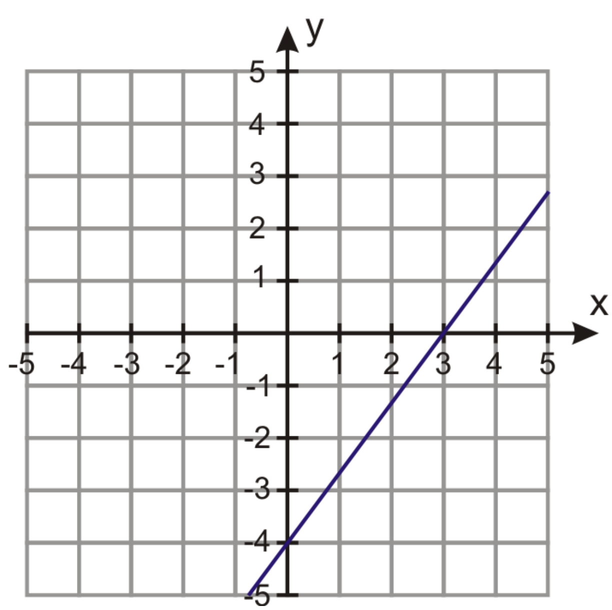


Now we want to apply this process in reverse—to start with the graph of the line and write the equation of the line in standard form.

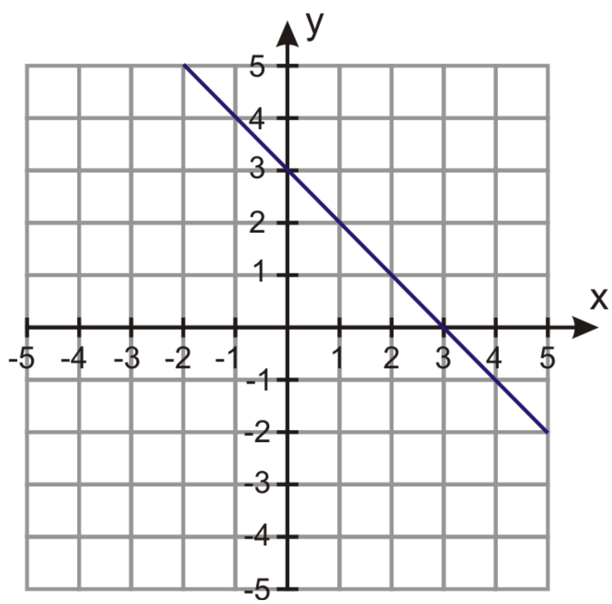
Example 3

Find the equation of each line and write it in standard form.

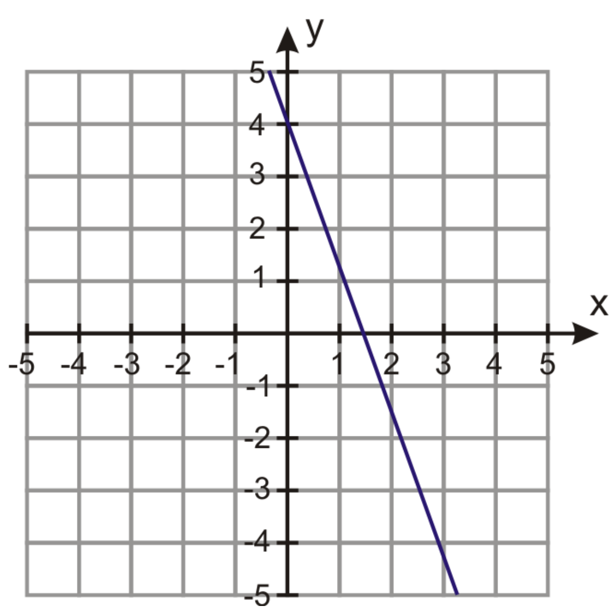
a)



b)



c)



Solution

a) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,-4) \Rightarrow y = -4$

We saw that in standard form $ax + by = c$: if we "cover up" the y term, we get $ax = c$, and if we "cover up" the x term, we get $by = c$.

So we need to find values for a and b so that we can plug in 3 for x and -4 for y and get the same value for c in both cases. This is like finding the least common multiple of the $x-$ and $y-$intercepts.

In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by –3 gives the same result:

$$(x = 3) \times 4 \Rightarrow 4x = 12 \quad \text{and} \quad (y = -4) \times (-3) \Rightarrow -3y = 12$$

Therefore, $a = 4, b = -3$ and $c = 12$ and **the equation in standard form is** $4x - 3y = 12$.

b) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,3) \Rightarrow y = 3$

The values of the intercept equations are already the same, so $a = 1, b = 1$ and $c = 3$. **The equation in standard form is** $x + y = 3$.

c) We see that the $x-$intercept is $\left(\frac{3}{2}, 0\right) \Rightarrow x = \frac{3}{2}$ and the $y-$intercept is $(0,4) \Rightarrow y = 4$

Let's multiply the $x-$intercept equation by $2 \Rightarrow 2x = 3$

Then we see we can multiply the $x-$intercept again by 4 and the $y-$intercept by 3, so we end up with $8x = 12$ and $3y = 12$.

The equation in standard form is $8x + 3y = 12$.

Solving Real-World Problems Using Linear Models in Point-Slope Form

Let's solve some word problems where we need to write the equation of a straight line in point-slope form.

Example 4

Marciel rented a moving truck for the day. Marciel only remembers that the rental truck company charges $40 per day and some number of cents per mile. Marciel drives 46 miles and the final amount of the bill (before tax) is $63. What is the amount per mile the truck rental company charges? Write an equation in point-slope form that describes this situation. How much would it cost to rent this truck if Marciel drove 220 miles?

Solution

Let's define our variables:

$x =$ distance in miles
$y =$ cost of the rental truck

Peter pays a flat fee of $40 for the day; this is the $y-$intercept.

He pays $63 for 46 miles; this is the coordinate point (46,63).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the coordinate point: $63 - y_0 = m(46 - x_0)$

Plug in the point (0, 40): $63 - 40 = m(46 - 0)$

Solve for the slope: $23 = 46m \rightarrow m = \frac{23}{46} = 0.5$

The slope is 0.5 dollars per mile, so the truck company charges 50 cents per mile ($0.5 = 50 cents). Plugging in the slope and the $y-$intercept, the equation of the line is $y = 0.5x + 40$.

To find out the cost of driving the truck 220 miles, we plug in $x = 220$ to get $y - 40 = 0.5(220) \Rightarrow y = \$150$.

Driving 220 miles would cost $150.

Example 5

Anne got a job selling window shades. She receives a monthly base salary and a $6 commission for each window shade she sells. At the end of the month she adds up sales and she figures out that she sold 200 window shades and made $2500. Write an equation in point-slope form that describes this situation. How much is Anne's monthly base salary?

Solution

Let's define our variables:

$x =$ number of window shades sold

$y =$ Annes earnings

We see that we are given the slope and a point on the line:

Nadia gets $6 for each shade, so the slope is 6.

She made $2500 when she sold 200 shades, so the point is (200, 2500).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the slope: $y - y_0 = 6(x - x_0)$

Plug in the point (200, 2500): $y - 2500 = 6(x - 200)$

To find Anne's base salary, we plug in $x = 0$ and get $y - 2500 = -1200 \Rightarrow y = \$1300$.

Anne's monthly base salary is $1300.

Solving Real-World Problems Using Linear Models in Standard Form

Here are two examples of real-world problems where the standard form of an equation is useful.

Example 6

Nadia buys fruit at her local farmer's market. This Saturday, oranges cost $2 per pound and cherries cost $3 per pound. She has $12 to spend on fruit. Write an equation in standard form that describes this situation. If she buys 4 pounds of oranges, how many pounds of cherries can she buy?

Solution

Let's define our variables:

$x =$ pounds of oranges
$y =$ pounds of cherries

The equation that describes this situation is $2x + 3y = 12$.

If she buys 4 pounds of oranges, we can plug $x = 4$ into the equation and solve for y:

$$2(4) + 3y = 12 \Rightarrow 3y = 12 - 8 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3}$$

Nadia can buy $1\frac{1}{3}$ pounds of cherries.

Example 7

Peter skateboards part of the way to school and walks the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If he skateboards for $\frac{1}{2}$ an hour, how long does he need to walk to get to school?

Solution

Let's define our variables:

$x =$ time Peter skateboards
$y =$ time Peter walks

The equation that describes this situation is: $7x + 3y = 6$

If Peter skateboards $\frac{1}{2}$ an hour, we can plug $x = 0.5$ into the equation and solve for y:

$$7(0.5) + 3y = 6 \Rightarrow 3y = 6 - 3.5 \Rightarrow 3y = 2.5 \Rightarrow y = \frac{5}{6}$$

Peter must walk $\frac{5}{6}$ of an hour.

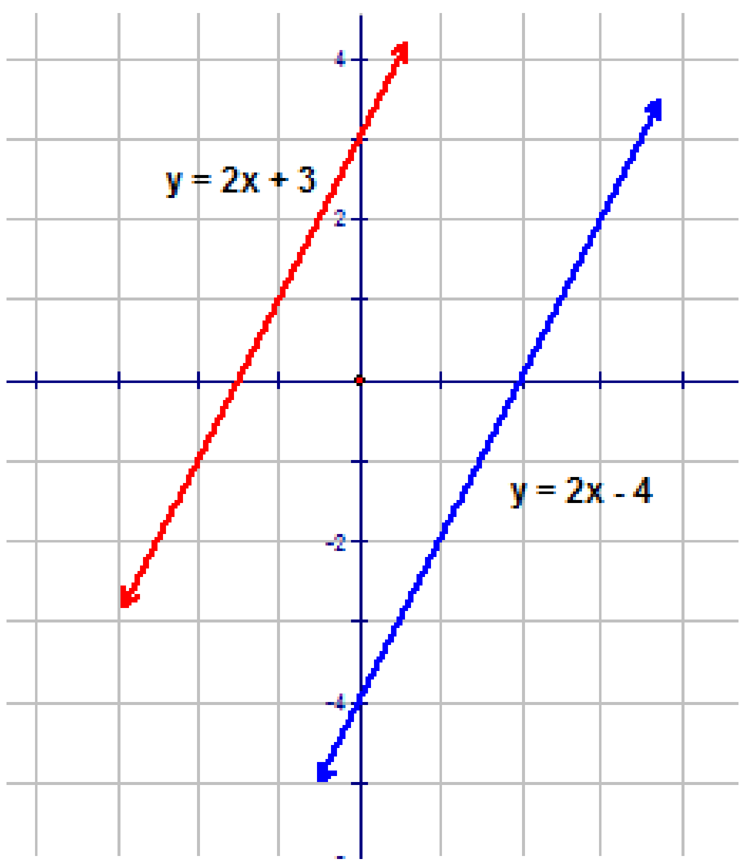
3.7 Equations of Parallel and Perpendicular Lines

Here you will learn about parallel and perpendicular lines and how to determine whether or not two lines are parallel or perpendicular using slope.

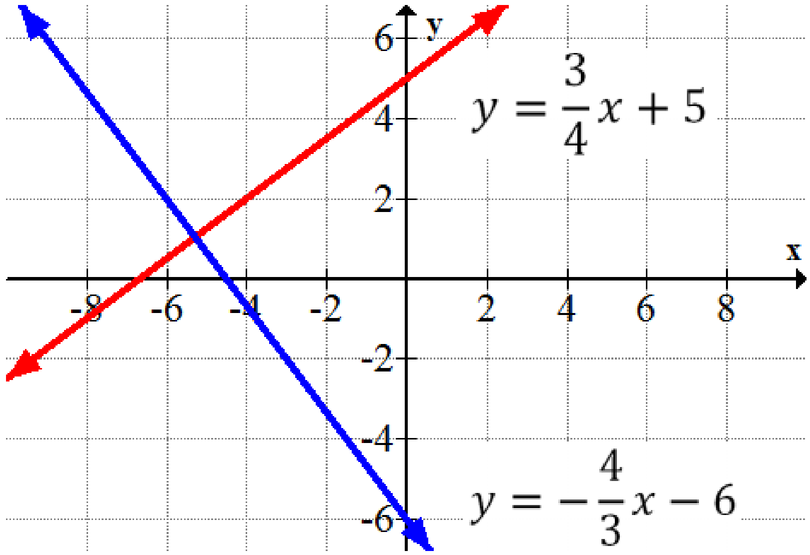
Can you write the equation for the line that passes through the point (–2, –3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

Guidance

Parallel lines are lines in the same plane that never intersect. Parallel lines maintain the **same slope**, or no slope (vertical lines) and the same distance from each other. The following graph shows two lines with the same slope. The slope of each line is 2. Notice that the lines are the same distance apart for the entire length of the lines. The lines will never intersect. The following lines are parallel.



Two lines in the same plane that intersect or cross each other at right angles are **perpendicular lines**. Perpendicular lines have **slopes that are opposite reciprocals**. The following graph shows two lines with slopes that are opposite reciprocals. The slope of one line is $\frac{3}{4}$ and the slope of the other line is $-\frac{4}{3}$. The product of the slopes is negative one. $\left(\frac{3}{4}\right)\left(\frac{-4}{3}\right)=\frac{-12}{12}=-1$. Notice that the lines intersect at a right angle. The lines are perpendicular lines.



You can use the relationship between the slopes of parallel lines and the slopes of perpendicular lines to write the equations of other lines.

Example A

Given the slopes of two lines, tell whether the lines are parallel, perpendicular or neither.

i) $m_1 = 4, m_2 = \frac{1}{4}$

ii) $m_1 = -3, m_2 = \frac{1}{3}$

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$

iv) $m_1 = -1, m_2 = 1$

v) $m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$

Solutions:

i) $m_1 = 4, m_2 = \frac{1}{4}$ The slopes are reciprocals but **not** opposite reciprocals. The lines are neither parallel nor perpendicular.

ii) $m_1 = -3, m_2 = \frac{1}{3}$ The slopes are reciprocals and are also opposite reciprocals. The lines are perpendicular.

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$ The slopes are the same. The fractions are equivalent. The lines are parallel.

iv) $m_1 = -1, m_2 = 1$ The slopes are reciprocals and are also opposite reciprocals. The lines are perpendicular.

v)$m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$ The slopes are not the same. The lines are neither parallel nor perpendicular.

Example B

Determine the equation of the line passing through the point (–4, 6) and parallel to the graph of $3x+2y-7=0$. Write the equation in standard form.

Solution:

If the equation of the line you are looking for is parallel to the given line, then the two lines have the same slope. Begin by expressing $3x+2y-7=0$ in slope-intercept form in order to find its slope.

$$\begin{aligned}
3x+2y-7&=0\\
3x-3x+2y-7&=0-3x\\
2y-7&=-3x\\
2y-7+7&=-3x+7\\
2y&=-3x+7\\
\frac{\cancel{2}y}{\cancel{2}}&=-\frac{3x}{2}+\frac{7}{2}\\
y&=-\frac{3}{2}x+\frac{7}{2}\\
&\ \updownarrow\\
y&=\ mx+b
\end{aligned}$$

The slope of the line is $-\frac{3}{2}$. The line passes through the point $(-4,6)$.

$$y-y_1=m(x-x_1)$$

Substitute the values into this equation.

$$\begin{aligned}
y-6&=\frac{-3}{2}(x--4)\\
y-6&=\frac{-3}{2}(x+4)\\
y-6&=\frac{-3x}{2}-\frac{12}{2}\\
2(y)-2(6)&=\cancel{2}\left(\frac{-3x}{\cancel{2}}\right)-\cancel{2}\left(\frac{12}{\cancel{2}}\right)\\
2y-12&=-3x-12\\
2y-12+12&=-3x-12+12\\
2y&=-3x\\
3x+2y&=-3x+3x\\
3x+2y&=0
\end{aligned}$$

The equation of the line is

$$\boxed{3x+2y=0}$$

Example C

Determine the equation of the line that passes through the point (6, –2) and is perpendicular to the graph of $3x=2y-4$. Write the equation in standard form.

Solution: Begin by writing the equation $3x=2y-4$ in slope-intercept form.

$$\begin{aligned}
3x &= 2y-4\\
2y-4 &= 3x\\
2y-4+4 &= 3x+4\\
2y &= 3x+4\\
\frac{\cancel{2}y}{\cancel{2}} &= \frac{3x}{2}+\frac{4}{2}\\
y &= \frac{3}{2}x+2\\
&\updownarrow\\
y &= mx+b
\end{aligned}$$

The slope of the given line is $\frac{3}{2}$. The slope of the perpendicular line is

$$\boxed{-\frac{2}{3}}$$

. The line passes through the point (6, –2).

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y--2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2x}{3}+\frac{12}{3}\\
3(y)+3(2) &= 3\left(-\frac{2x}{3}\right)+3\left(\frac{12}{3}\right)\\
3(y)+3(2) &= \cancel{3}\left(-\frac{2x}{\cancel{3}}\right)+\cancel{3}\left(\frac{12}{\cancel{3}}\right)\\
3y+6 &= -2x+12\\
3y+6-12 &= -2x+12-12\\
3y-6 &= -2x\\
2x+3y-6 &= -2x+2x\\
2x+3y-6 &= 0
\end{aligned}$$

The equation of the line is

$$\boxed{2x+3y-6=0}$$

.

Concept Problem Revisited

Can you write the equation for the line that passes through the point (–2, –3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the parallel line is the same as the slope of the given line.

$$\begin{aligned} y+2x &= 8 \\ y+2x-2x &= -2x+8 \\ y &= -2x+8 \end{aligned}$$

The slope of the given line is –2. The slope of the parallel line is also

$$\boxed{-2}$$

.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y--3 &= -2(x--2) \\ y+3 &= -2(x+2) \\ y+3 &= -2x-4 \\ y+3 &= -2x-4 \\ 2x+y+3 &= -2x+2x-4 \\ 2x+y+3 &= -4 \\ 2x+y+3+4 &= -4+4 \\ 2x+y+7 &= 0 \end{aligned}$$

The equation of the line is

$$\boxed{2x+y+7=0}$$

Vocabulary

Parallel Lines

Parallel lines are lines in the same plane that have the same slope. The lines never intersect and always maintain the same distance apart.

Perpendicular Lines

Perpendicular lines are lines in the same plane that intersect each other at right angles. The slopes of perpendicular lines are opposite reciprocals. The product of the slopes of two perpendicular lines is –1.

Guided Practice

Determine whether the lines that pass through the two pairs of points are parallel, perpendicular or neither parallel nor perpendicular.

1. (–2, 8), (3, 7) and (4, 3), (9, 2)

2. (2, 5), (8, 7) and (–3, 1), (–2, –2)

3. (4, 6), (–3, –1) and (6, –3), (4, 5)

4. Write the equation for the line that passes through the point (–3, 6) and is perpendicular to the graph of $3x = 5y + 6$. Write the equation of the line in slope-intercept form.

Answers:

1. (–2, 8), (3, 7) and (4, 3), (9, 2)

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Use the formula to calculate the slopes}$$

$$m = \frac{7-8}{3--2} \qquad m = \frac{2-3}{9-4} \qquad \text{Calculate the slopes for each pair of points}$$

$$m = \frac{7-8}{3+2} \qquad \boxed{m = \frac{-1}{5}}$$

$$\boxed{m = \frac{-1}{5}}$$

The slopes of the lines are the same. The lines are parallel.

2. (2, 5), (8, 7) and (–3, 1), (–2, –2)

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Use the formula to calculate the slopes}$$

$$m = \frac{7-5}{8-2} \qquad m = \frac{-2-1}{-2--3} \qquad \text{Calculate the slopes for each pair of points}$$

$$m = \frac{2}{6} \qquad m = \frac{-2-1}{-2+3}$$

$$\boxed{m = \frac{1}{3}} \qquad \boxed{m = \frac{-3}{1}}$$

The slopes of the lines are opposite reciprocals. The lines are perpendicular.

3. (4, 6), (–3, –1) and (6, –3), (4, 5)

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Use the formula to calculate the slopes}$$

$$m = \frac{-1-6}{-3-4} \qquad m = \frac{5--3}{4-6} \qquad \text{Calculate the slopes for each pair of points}$$

$$m = \frac{-7}{-7} \qquad m = \frac{5+3}{4-6}$$

$$\boxed{m = 1} \qquad m = \frac{8}{-2}$$

$$\boxed{m = -4}$$

The lines are neither parallel nor perpendicular.

4. Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the perpendicular line is the opposite reciprocal.

$$
\begin{aligned}
3x &= 5y+6 \\
5y+6 &= 3x \\
5y+6-6 &= 3x-6 \\
5y &= 3x-6 \\
\frac{5y}{5} &= \frac{3x}{5}-\frac{6}{5} \\
\frac{\cancel{5}y}{\cancel{5}} &= \frac{3x}{5}-\frac{6}{5} \\
y &= \frac{3}{5}x-\frac{6}{5}
\end{aligned}
$$

The slope of the given line is $\frac{3}{5}$. The slope of the perpendicular line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the perpendicular line that passes through the point (–3, 6) is:

$$
\begin{aligned}
y &= mx+b \\
6 &= -\frac{5}{3}(-3)+b \\
6 &= -\frac{5}{\cancel{3}}\left(\cancel{-}\overset{-1}{\cancel{3}}\right)+b \\
6 &= 5+b \\
6-5 &= 5-5+b \\
1 &= b
\end{aligned}
$$

The y-intercept is (0, 1) and the slope of the line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the line is

$$\boxed{y=-\frac{5}{3}x+1}$$

.

Practice

For each pair of given equations, determine if the lines are parallel, perpendicular or neither.

1. $y=2x-5$ and $y=2x+3$
2. $y=\frac{1}{3}x+5$ and $y=-3x-5$
3. $x=8$ and $x=-2$
4. $y=4x+7$ and $y=-4x-7$

5. $y = -x - 3$ and $y = x + 6$
6. $3y = 9x + 8$ and $y = 3x - 4$

Determine the equation of the line satisfying the following conditions:

7. through the point (5, –6) and parallel to the line $y = 5x + 4$
8. through the point (–1, 7) and perpendicular to the line $y = -4x + 5$
9. containing the point (–1, –5) and parallel to $3x + 2y = 9$
10. containing the point (0, –6) and perpendicular to $6x - 3y + 8 = 0$
11. through the point (2, 4) and perpendicular to the line $y = -\frac{1}{2}x + 3$
12. containing the point (–1, 5) and parallel to $x + 5y = 3$
13. through the point (0, 4) and perpendicular to the line $2x - 5y + 1 = 0$

If $D(4, -1), E(-4, 5)$ and $F(3, 6)$ are the vertices of ΔDEF determine:

14. the equation of the line through D and parallel to EF.
15. the equation of the line containing the altitude from D to EF (the line perpendicular to EF that contains D).