CHAPTER 4 Chapter 4: Functions

Chapter Outline

4.1 Function Notation

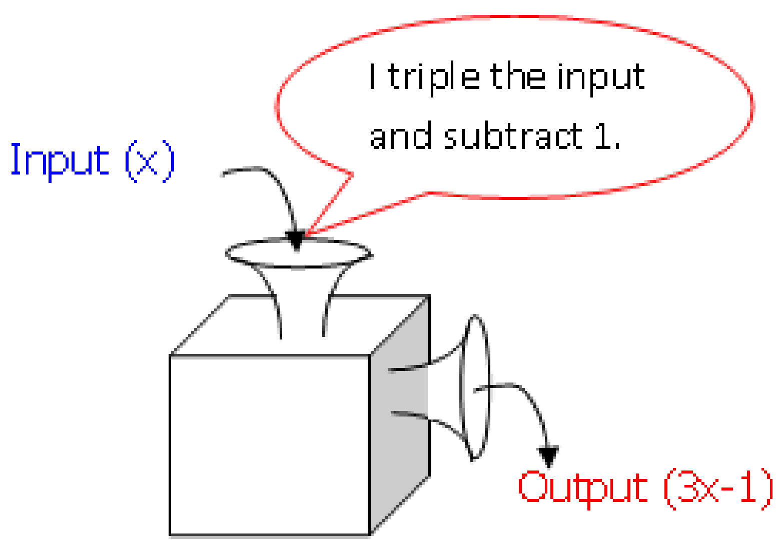
Here you'll learn how to use function notation when working with functions.

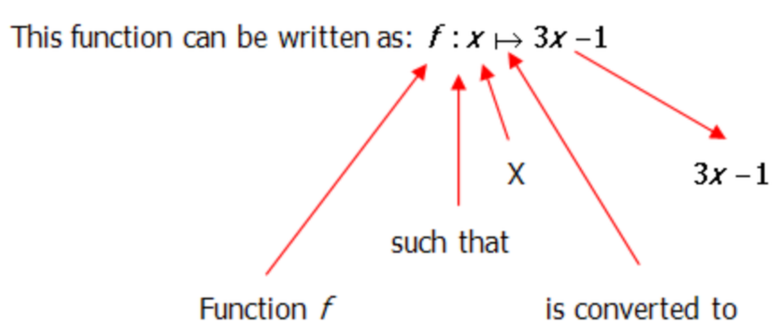
Suppose the value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$.

- Can you determine the value of $V(4)$ and explain what the solution means in the context of this problem?
- Can you determine the value of t when $V(t) = 525$ and explain what this situation represents?
- What was the original cost of the digital camera?

Guidance

A function machine shows how a function responds to an input. If I triple the input and subtract one, the machine will convert x into $3x - 1$. So, for example, if the function is named f, and 3 is fed into the machine, $3(3) - 1 = 8$ comes out.





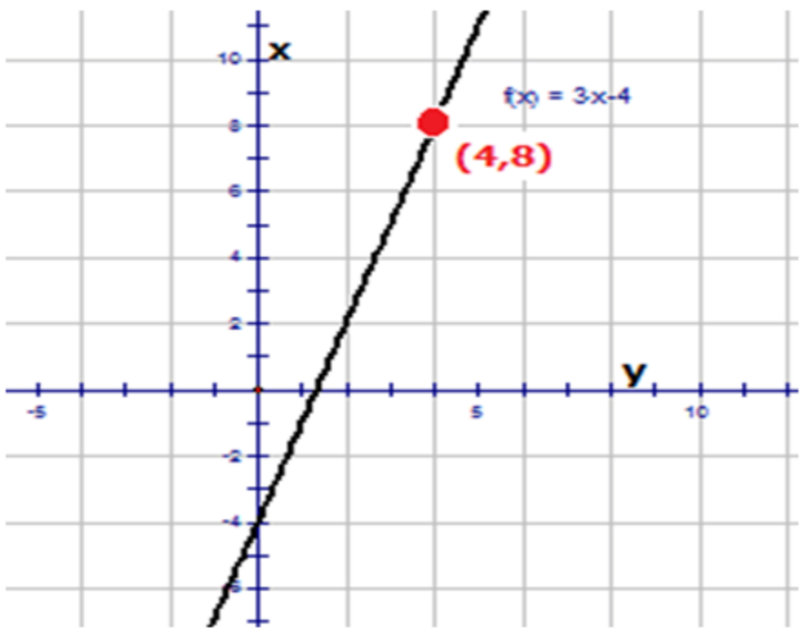
When naming a function the symbol $f(x)$ is often used. The symbol $f(x)$ is pronounced as "f of x." This means that the equation is a function that is written in terms of the variable x. An example of such a function is $f(x) = 3x+4$. Functions can also be written using a letter other than f and a variable other than x. For example, $v(t) = 2t^2 - 5$ and $d(h) = 4h-3$. In addition to representing a function as an equation, you can also represent a function:

- As a graph
- As ordered pairs
- As a table of values
- As an arrow or mapping diagram

When a function is represented as an equation, an ordered pair can be determined by evaluating various values of the assigned variable. Suppose $f(x) = 3x-4$. To calculate $f(4)$, substitute:

$$\begin{aligned} f(4) &= 3(4)-4 \\ f(4) &= 12-4 \\ f(4) &= 8 \end{aligned}$$

Graphically, if $f(4) = 8$, this means that the point (4, 8) is a point on the graph of the line.



Example A

If $f(x) = x^2 + 2x + 5$ find.

a) $f(2)$

b) $f(-7)$

c) $f(1.4)$

Solution:

To determine the value of the function for the assigned values of the variable, substitute the values into the function.

$$\begin{array}{lll} f(x) = x^2 + 2x + 5 & f(x) = x^2 + 2x + 5 & f(x) = x^2 + 2x + 5 \\ \downarrow \quad \downarrow \quad \searrow & \downarrow \quad \downarrow \quad \searrow & \downarrow \quad \downarrow \quad \searrow \\ f(2) = (2)^2 + 2(2) + 5 & f(-7) = (-7)^2 + 2(-7) + 5 & f(1.4) = (1.4)^2 + 2(1.4) + 5 \\ f(2) = 4 + 4 + 5 & f(-7) = 49 - 14 + 5 & f(1.4) = 1.96 + 2.8 + 5 \\ \boxed{f(2) = 13} & \boxed{f(-7) = 40} & \boxed{f(1.4) = 9.76} \end{array}$$

Example B

Functions can also be represented as mapping rules. If $g : x \to 5 - 2x$ find the following in simplest form:

a) $g(y)$

b) $g(y-3)$

c) $g(2y)$

Solution:

a) $g(y) = 5 - 2y$

b) $g(y-3) = 5 - 2(y-3) = 5 - 2y + 6 = 11 - 2y$

c) $g(2y) = 5 - 2(2y) = 5 - 4y$

Example C

Let $P(a) = \frac{2a-3}{a+2}$.

a) Evaluate

i) $P(0)$
ii) $P(1)$
iii) $P\left(-\frac{1}{2}\right)$

b) Find a value of 'a' where $P(a)$ does not exist.

c) Find $P(a-2)$ in simplest form

d) Find 'a' if $P(a) = -5$

Solution:

a)

$$P(a)=\frac{2a-3}{a+2}$$
$$P(0)=\frac{2(0)-3}{(0)+2}$$
$$\boxed{P(0)=\frac{-3}{2}}$$

$$P(a)=\frac{2a-3}{a+2}$$
$$P(1)=\frac{2(1)-3}{(1)+2}$$
$$P(1)=\frac{2-3}{1+2}$$
$$\boxed{P(1)=\frac{-1}{3}}$$

$$P(a)=\frac{2a-3}{a+2}$$
$$P\left(-\frac{1}{2}\right)=\frac{2\left(-\frac{1}{2}\right)-3}{\left(-\frac{1}{2}\right)+2}$$
$$P\left(-\frac{1}{2}\right)=\frac{{}^{1}\not{2}\left(-\frac{1}{\not{2}}\right)-3}{-\frac{1}{2}+\frac{4}{2}}$$
$$P\left(-\frac{1}{2}\right)=\frac{-1-3}{\frac{3}{2}}$$
$$P\left(-\frac{1}{2}\right)=-4\div\frac{3}{2}$$
$$P\left(-\frac{1}{2}\right)=-4\left(\frac{2}{3}\right)$$
$$\boxed{P\left(-\frac{1}{2}\right)}=\frac{-8}{3}$$

b) The function will not exist if the denominator equals zero because division by zero is undefined.

$$a+2=0$$
$$a+2-2=0-2$$
$$\boxed{a=-2}$$

Therefore, if $a=-2$, then $P(a)=\frac{2a-3}{a+2}$ does not exist.

c)

$$P(a)=\frac{2a-3}{a+2}$$
$$P(a-2)=\frac{2(a-2)-3}{(a-2)+2} \qquad \text{Substitue } a-2 \text{ for } a$$
$$P(a-2)=\frac{2a-4-3}{a-2+2} \qquad \text{Remove parentheses}$$
$$P(a-2)=\frac{2a-7}{a} \qquad \text{Combine like terms}$$
$$P(a-2)=\frac{2\not{a}}{\not{a}}-\frac{7}{a} \qquad \text{Express the fraction as two separate fractions and reduce.}$$
$$\boxed{P(a-2)=2-\frac{7}{a}}$$

d)

$$
\begin{aligned}
P(a) &= \frac{2a-3}{a+2} \\
-5 &= \frac{2a-3}{a+2} && \text{Let } P(a) = -5 \\
-5(a+2) &= \left(\frac{2a-3}{a+2}\right)(a+2) && \text{Multiply both sides by } (a+2) \\
-5a-10 &= \left(\frac{2a-3}{\cancel{a+2}}\right)(\cancel{a+2}) && \text{Simplify} \\
-5a-10 &= 2a-3 && \text{Solve the linear equation} \\
-5a-10-2a &= 2a-2a-3 && \text{Move } 2a \text{ to the left by subtracting} \\
-7a-10 &= -3 && \text{Simplify} \\
-7a-10+10 &= -3+10 && \text{Move 10 to the right side by addition} \\
-7a &= 7 && \text{Simplify} \\
\frac{-7a}{-7} &= \frac{7}{-7} && \text{Divide both sides by -7 to solve for } a. \\
\boxed{a = -1}
\end{aligned}
$$

Concept Problem Revisited

The value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$

- Determine the value of $V(4)$ and explain what the solution means to this problem.
- Determine the value of t when $V(t) = 525$ and explain what this situation represents.
- What was the original cost of the digital camera?

Solution:

- The camera is valued at $675, 4 years after it was purchased.

$$
\begin{aligned}
V(t) &= 875 - 50t \\
V(4) &= 875 - 50(4) \\
V(4) &= 875 - 200 \\
\boxed{V(4) = \$675}
\end{aligned}
$$

- The digital camera has a value of $525, 7 years after it was purchased.

$$
\begin{aligned}
V(t) &= 875 - 50t && \text{Let } V(t) = 525 \\
525 &= 875 - 50t && \text{Solve the equation} \\
525 - 875 &= 875 - 875 - 50t \\
-350 &= -50t \\
\frac{-350}{-50} &= \frac{-50t}{-50} \\
\boxed{7 = t}
\end{aligned}
$$

- The original cost of the camera was \$875.

$$\begin{aligned} V(t) &= 875 - 50t && \text{Let } t = 0. \\ V(0) &= 875 - 50(0) \\ V(0) &= 875 - 0 \\ &\boxed{V(0) = \$875} \end{aligned}$$

Vocabulary

Function

A ***function*** is a set of ordered pairs (x, y) that shows a relationship where there is only one output for every input. In other words, for every value of x, there is only one value for y.

Guided Practice

1. If $f(x) = 3x^2 - 4x + 6$ find:

i) $f(-3)$
ii) $f(a-2)$

2. If $f(m) = \frac{m+3}{2m-5}$ find 'm' if $f(m) = \frac{12}{13}$

3. The emergency brake cable in a truck parked on a steep hill breaks and the truck rolls down the hill. The distance in feet, d, that the truck rolls is represented by the function $d = f(t) = 0.5t^2$.

i) How far will the truck roll after 9 seconds?
ii) How long will it take the truck to hit a tree which is at the bottom of the hill 600 feet away? *Round your answer to the nearest second.*

Answers:

1. $f(x) = 3x^2 - 4x + 6$

i)

$$\begin{aligned} f(x) &= 3x^2 - 4x + 6 && \text{Substitute } (-3) \text{ for } x \text{ in the function.} \\ f(-3) &= 3(-3)^2 - 4(-3) + 6 && \text{Perform the indicated operations.} \\ f(-3) &= 3(9) + 12 + 6 && \text{Simplify} \\ f(-3) &= 27 + 12 + 6 \\ f(-3) &= 45 \\ &\boxed{f(-3) = 45} \end{aligned}$$

ii)

$$f(x) = 3x^2 - 4x + 6$$
$$f(a-2) = 3(a-2)^2 - 4(a-2) + 6$$ Write $(a-2)^2$ in expanded form.
$$f(a-2) = 3(a-2)(a-2) - 4(a-2) + 6$$ Perform the indicated operations.
$$f(a-2) = (3a-6)(a-2) - 4(a-2) + 6$$
$$f(a-2) = 3a^2 - 6a - 6a + 12 - 4a + 8 + 6$$ Simplify
$$f(a-2) = 3a^2 - 16a + 26$$
$$\boxed{f(a-2) = 3a^2 - 16a + 26}$$

2.

$$f(m) = \frac{m+3}{2m-5}$$
$$\frac{12}{13} = \frac{m+3}{2m-5}$$ Solve the equation for m.
$$(13)(2m-5)\frac{12}{13} = (13)(2m-5)\frac{m+3}{2m-5}$$
$$\cancel{(13)}(2m-5)\frac{12}{\cancel{13}} = (13)\cancel{(2m-5)}\frac{m+3}{\cancel{2m-5}}$$
$$(2m-5)12 = (13)m + 3$$
$$24m - 60 = 13m + 39$$
$$24m - 60 + 60 = 13m + 39 + 60$$
$$24m = 13m + 99$$
$$24m - 13m = 13m - 13m + 99$$
$$11m = 99$$
$$\frac{11m}{11} = \frac{99}{11}$$
$$\frac{\cancel{11}m}{\cancel{11}} = \frac{\cancelto{9}{99}}{\cancel{11}}$$
$$\boxed{m = 9}$$

3. $d = f(t) = 0.5^2$

i)

$$d = f(t) = 0.5^2$$ Substitute 9 for t.
$$f(9) = 0.5(9)^2$$ Perform the indicated operations.
$$f(9) = 0.5(81)$$
$$\boxed{f(9) = 40.5\ feet}$$

After 9 seconds, the truck will roll 40.5 feet.
ii)

$$d = f(t) = 0.5t^2 \qquad \text{Substitute 600 for } d.$$
$$600 = 0.5t^2 \qquad \text{Solve for } t.$$
$$\frac{600}{0.5} = \frac{0.5t^2}{0.5}$$
$$\frac{\overset{1200}{\cancel{600}}}{\cancel{0.5}} = \frac{\cancel{0.5}t^2}{\cancel{0.5}}$$
$$1200 = t^2$$
$$\sqrt{1200} = \sqrt{t^2}$$
$$\boxed{34.64 \ seconds \approx t}$$

The truck will hit the tree in approximately 35 seconds.

Practice

If $g(x) = 4x^2 - 3x + 2$, find expressions for the following:

1. $g(a)$
2. $g(a-1)$
3. $g(a+2)$
4. $g(2a)$
5. $g(-a)$

If $f(y) = 5y - 3$, determine the value of 'y' when:

6. $f(y) = 7$
7. $f(y) = -1$
8. $f(y) = -3$
9. $f(y) = 6$
10. $f(y) = -8$

The value of a Bobby Orr rookie card n years after its purchase is $V(n) = 520 + 28n$.

11. Determine the value of $V(6)$ and explain what the solution means.
12. Determine the value of n when $V(n) = 744$ and explain what this situation represents.
13. Determine the original price of the card.

Let $f(x) = \frac{3x}{x+2}$.

14. When is $f(x)$ undefined?
15. For what value of x does $f(x) = 2.4$?

4.2 Domain and Range

Here you'll learn how to find the domain and range of a relation.

Joseph drove from his summer home to his place of work. To avoid the road construction, Joseph decided to travel the gravel road. After driving for 20 minutes he was 62 miles away from work and after driving for 40 minutes he was 52 miles away from work. Represent the problem on a graph and write a suitable domain and range for the situation.

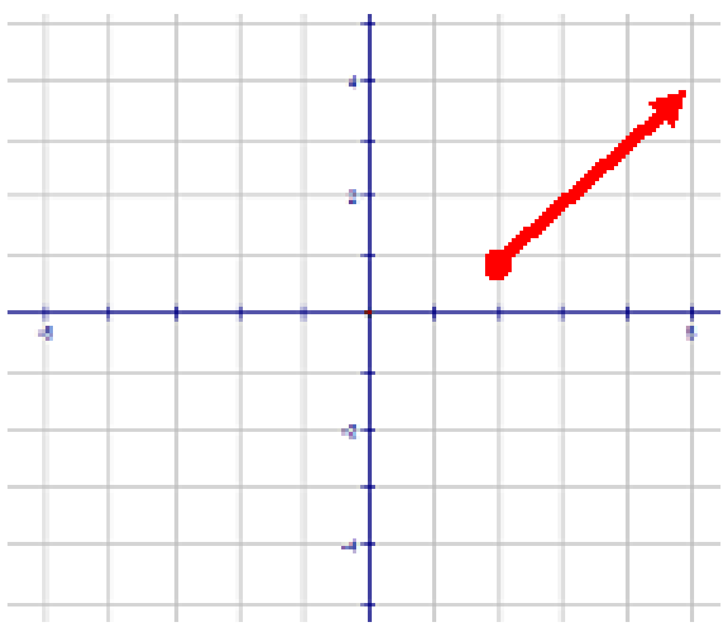
Guidance

The domain of a relation is the set of possible values that 'x' may have. The range of a relation is the set of possible values that 'y' may have. You can write the domain and range of a relation using interval notation and with respect to the number system to which it belongs. Remember:

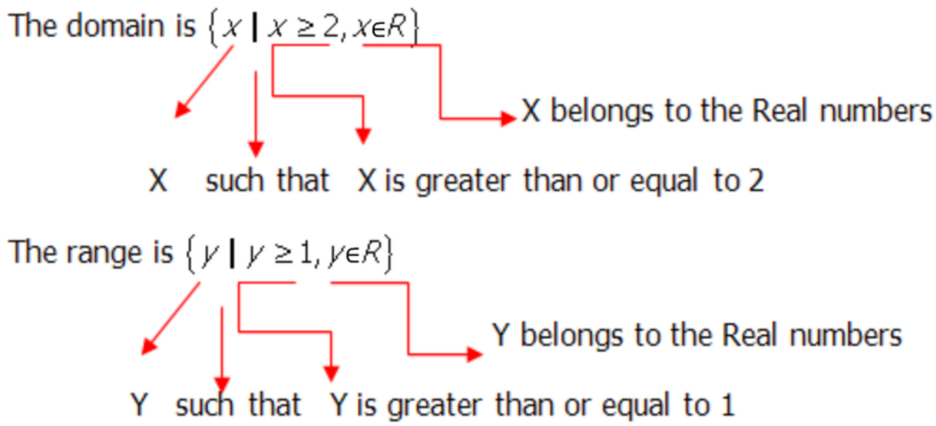
- $Z(\text{integers}) = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $R(\text{real numbers}) = \{\text{all rational and irrational numbers}\}$.

These number systems are very important when the domain and range of a relation are described using interval notation.

A relation is said to be discrete if there are a finite number of data points on its graph. Graphs of discrete relations appear as dots. A relation is said to be continuous if its graph is an unbroken curve with no "holes" or "gaps." The graph of a continuous relation is represented by a line or a curve like the one below. Note that it is possible for a relation to be neither discrete nor continuous.



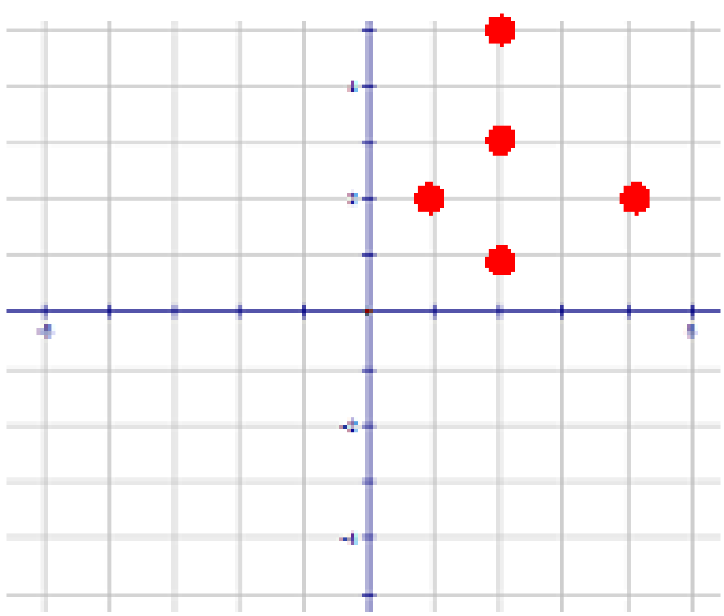
The relation is a straight line that that begins at the point (2, 1). The fact that the points on the line are connected indicates that the relation is continuous. The domain and the range can be written in interval notation, as shown below:



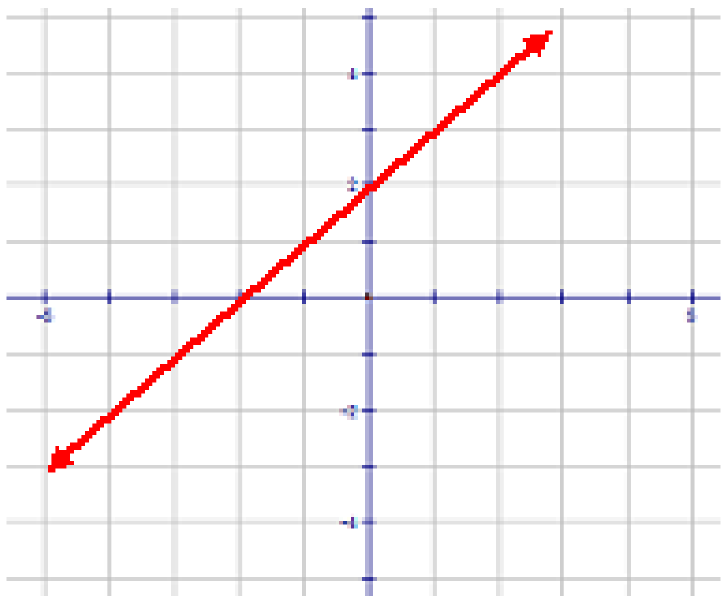
Example A

Which relations are discrete? Which relations are continuous? For each relation, find the domain and range.

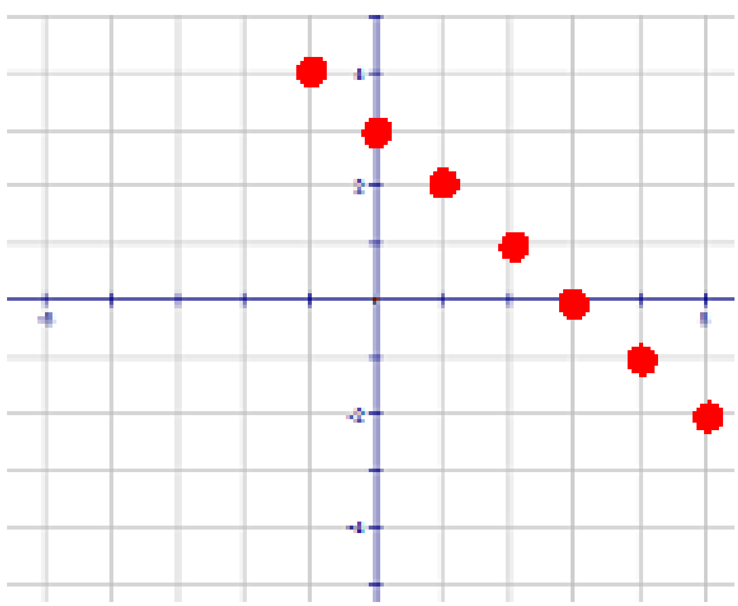
(i)



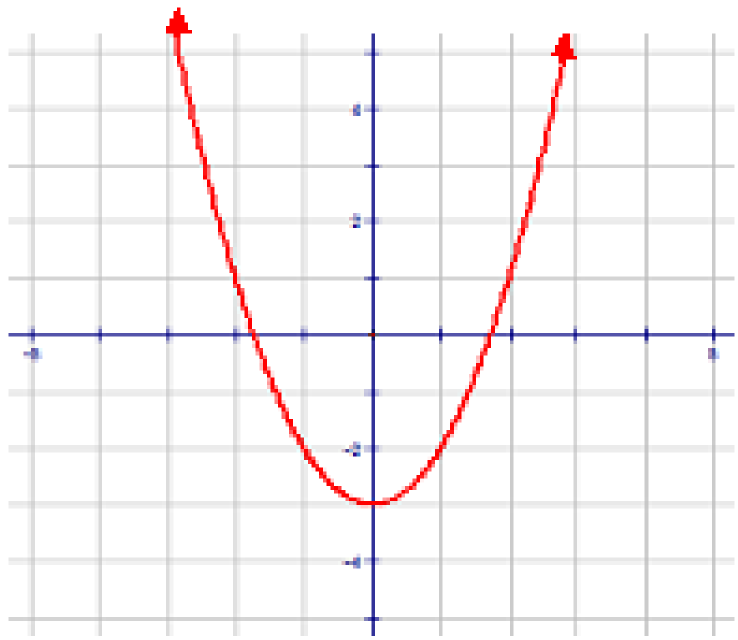
(ii)



(iii)



(iv)



Solution:

(i) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{1,2,4\}$. The range is $\{1,2,3,5\}$

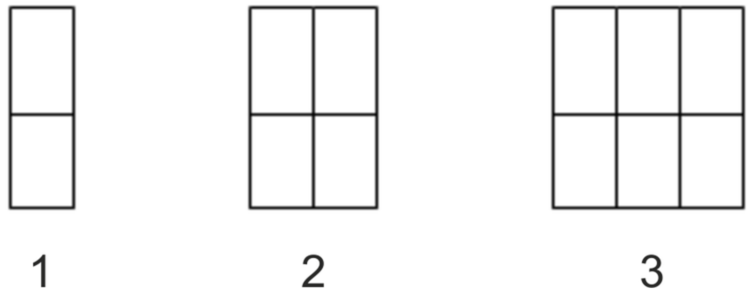
(ii) The graph appears as a straight line. Therefore, the relation is continuous. $D = \{x|x \in R\} \quad R = \{y|y \in R\}$

(iii) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{-1,0,1,2,3,4,5\}$. The range is $\{-2,-1,0,1,2,3,4\}$

(iv) The graph appears as a curve. Therefore, the relation is continuous. $D = \{x|x \in R\} \quad R = \{y|y \geq -3, y \in R\}$

Example B

Whether a relation is discrete, continuous, or neither can often be determined without a graph. The domain and range can be determined without a graph as well. Examine the following toothpick pattern.



Complete the table below to determine the number of toothpicks needed for the pattern.

TABLE 4.1:

Pattern number (n)	1	2	3	4	5	...	n	...	200
Number of toothpicks (t)									

Is the data continuous or discrete? Why?

What is the domain?

What is the range?

Solution:

TABLE 4.2:

Pattern number (n)	1	2	3	4	5	...	n	...	200
Number of toothpicks (t)	7	12	17	22	27		$5n+2$		1002

The number of toothpicks in any pattern number is the result of multiplying the pattern number by 5 and adding 2 to the product.

The number of toothpicks in pattern number 200 is:

$$\begin{aligned}
t &= 5n+2 \\
t &= 5(200)+2 \\
t &= 1000+2 \\
t &= 1002
\end{aligned}$$

The data must be discrete. The graph would be dots representing the pattern number and the number of toothpicks. It is impossible to have a pattern number or a number of toothpicks that are not natural numbers. Therefore, the points would not be joined.

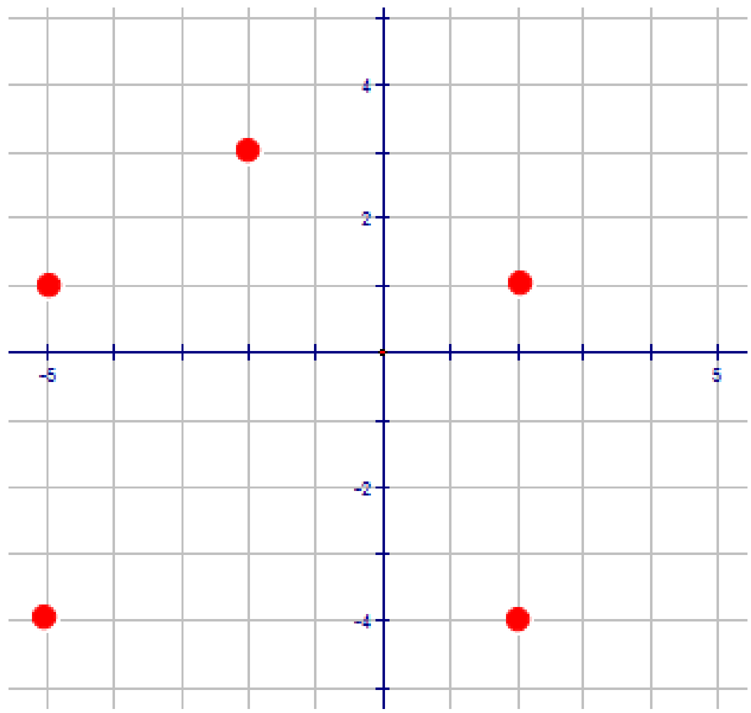
The domain and range are:

$$D = \{x|x \in N\} \quad R = \{y|y = 5x+2, x \in N\}$$

If the range is written in terms of a function, then the number system to which x belongs must be designated in the range.

Example C

Can you state the domain and the range of the following relation?



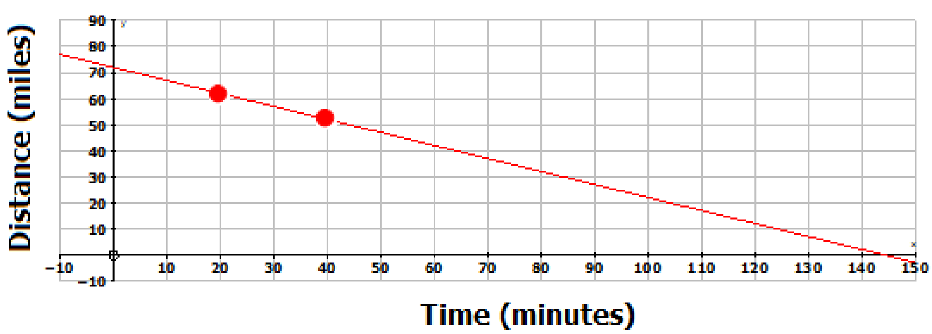
Solution:

The points indicated on the graph are $\{(-5,-4),(-5,1),(-2,3),(2,1),(2,-4)\}$

The domain is $\{-5,-2,2\}$ and the range is $\{-4,1,3\}$.

Concept Problem Revisited

Joseph drove from his summer home to his place of work. To avoid the road construction, Joseph decided to travel the gravel road. After driving for 20 minutes he was 62 miles away from work and after driving for 40 minutes he was 52 miles away from work. Represent the problem on a graph and write a suitable domain and range for the situation.



To represent the problem on a graph, plot the points (20, 62) and (40, 52). The points can be joined with a straight line since the data is continuous. The distance traveled changes continuously as the time driving changes. The y-intercept represents the distance from Joseph's summer home to his place of work. This distance is approximately 72 miles. The x-intercept represents the time it took Joseph to drive from his summer home to work. This time is approximately 145 minutes.

Time cannot be a negative quantity. Therefore, the smallest value for the number of minutes would have to be zero. This represents the time Joseph began his trip. A suitable domain for this problem is $D = \{x|0 \leq x \leq 145, x \in R\}$

The distance from his summer home to work cannot be a negative quantity. This distance is represented on the y-axis as the y-intercept and is the distance before he begins to drive. A suitable range for the problem is $R = \{y|0 \leq y \leq 72, y \in R\}$

The domain and range often depend on the quantities presented in the problem. In the above problem, the quantities of time and distance could not be negative. As a result, the values of the domain and the range had to be positive.

Vocabulary

Continuous

A relation is said to be ***continuous*** if it is an unbroken curve with no "holes" or "gaps".

Discrete

A relation is said to be ***discrete*** if there are a finite number of data points on its graph. Graphs of ***discrete*** relations appear as dots.

Domain

The ***domain*** of a relation is the set of possible values that 'x' may have.

Range

The ***range*** of a relation is the set of possible values that 'y' may have.

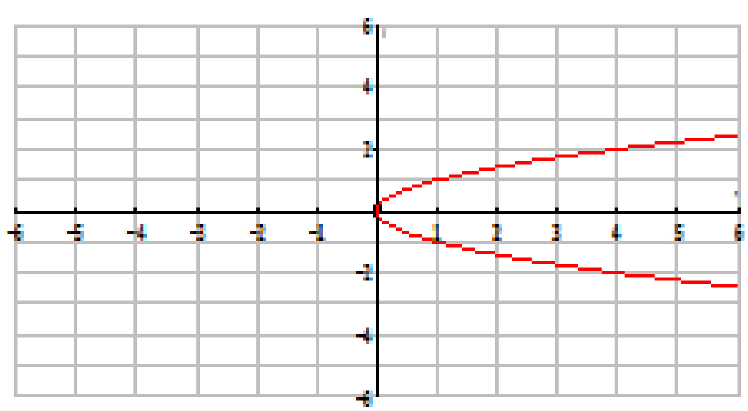
Coordinates

The ***coordinates*** are the ordered pair (x, y) that represents a point on the Cartesian plane.

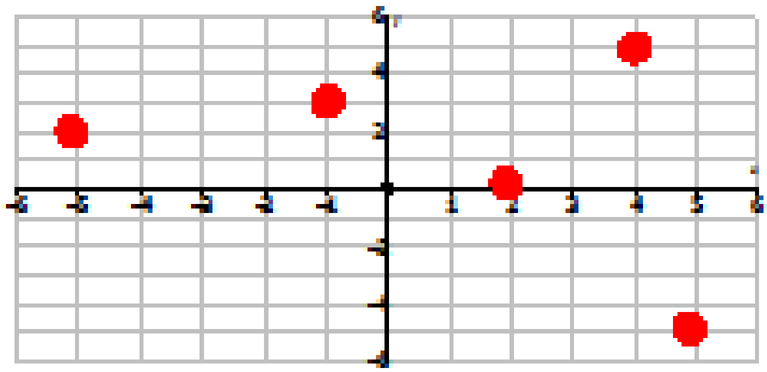
Guided Practice

1. Which relation is discrete? Which relation is continuous?

(i)

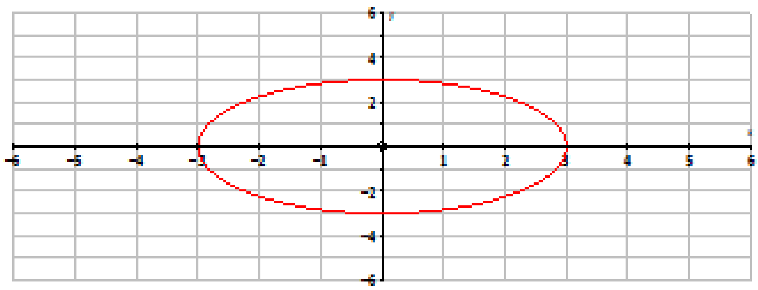


(ii)

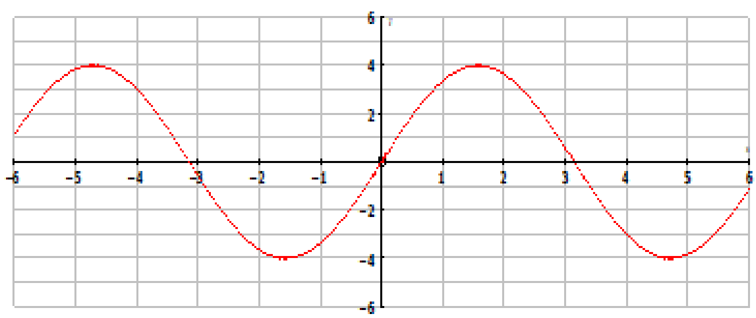


2. State the domain and the range for each of the following relations:

(i)



(ii)



3. A computer salesman's wage consists of a monthly salary of $200 plus a bonus of $100 for each computer sold.

(a) Complete the following table of values:

TABLE 4.3:

Number of computers sold	**0**	**2**	**5**	**10**	**18**
Wages in dollars for the month ($)					

(b) Sketch the graph to represent the monthly salary ($), against the number (N), of computers sold.
(c) Use the graph to write a suitable domain and range for the problem.

Answers:

1. (i) The graph clearly shows that the points are joined. Therefore the data is continuous.

(ii) The graph shows the plotted points as dots that are not joined. Therefore the data is discrete.

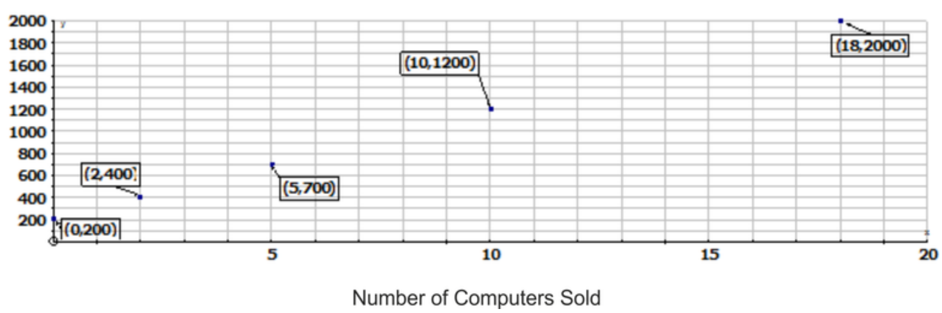
2. (i) The domain represents the values of 'x'. $D = \{x | -3 \leq x \leq 3, x \in R\}$

The range represents the values of 'y'. $R = \{y | -3 \leq y \leq 3, y \in R\}$
(ii) $D = \{x | x \in R\}$
$R = \{y | -4 \leq y \leq 4, y \in R\}$

3.

TABLE 4.4:

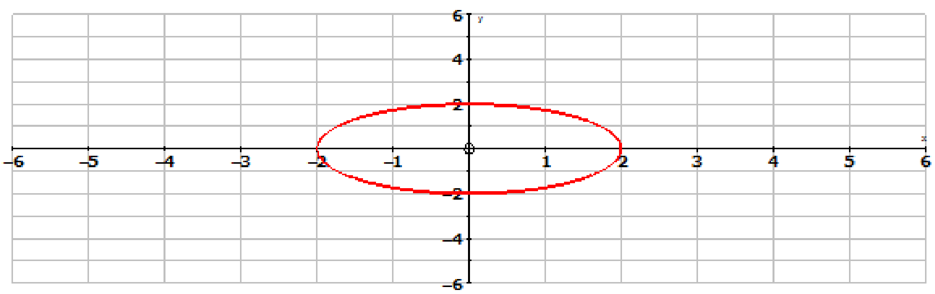
Number of computers sold	**0**	**2**	**5**	**10**	**18**
Wages in dollars for the month ($)	**$200**	**$400**	**$700**	**$1200**	**$2000**



(c) The graph shows that the data is discrete. (The salesman can't sell a portion of a computer, so the data points can't be connected.) The number of computers sold and must be whole numbers. The wages must be natural numbers.
A suitable domain is $D = \{x | x \geq 0, x \in W\}$
A suitable domain is $R = \{y | y = 200 + 100x, x \in N\}$

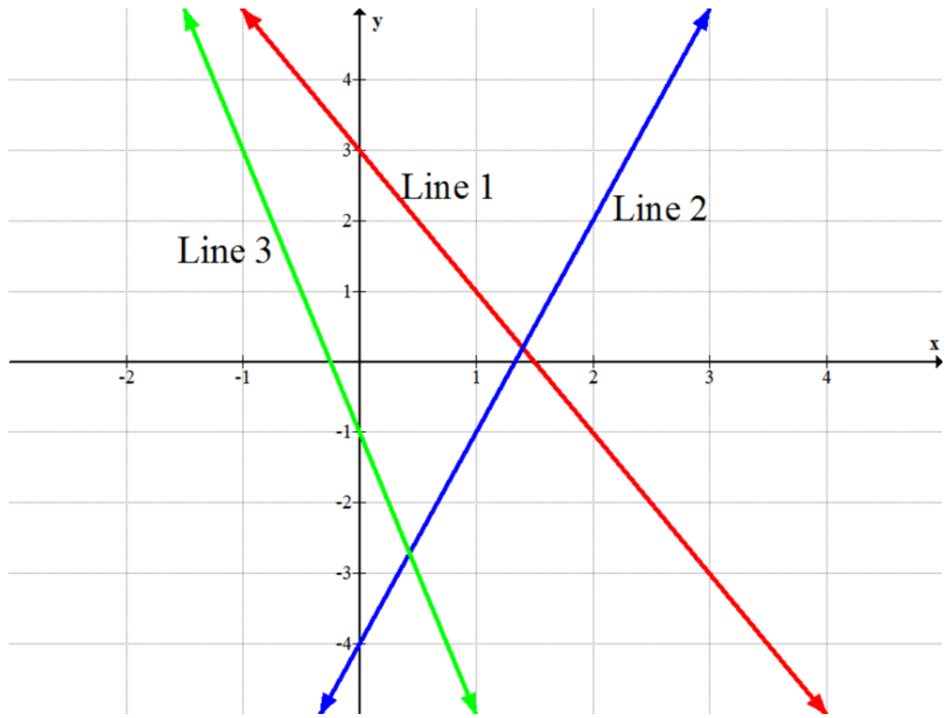
Practice

Use the graph below for #1 and #2.



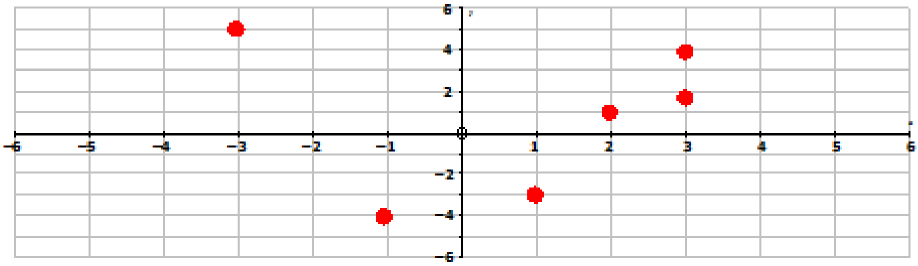
1. Is the relation discrete, continuous, or neither?
2. Find the domain and range for the relation.

Use the graph below for #3 and #4.



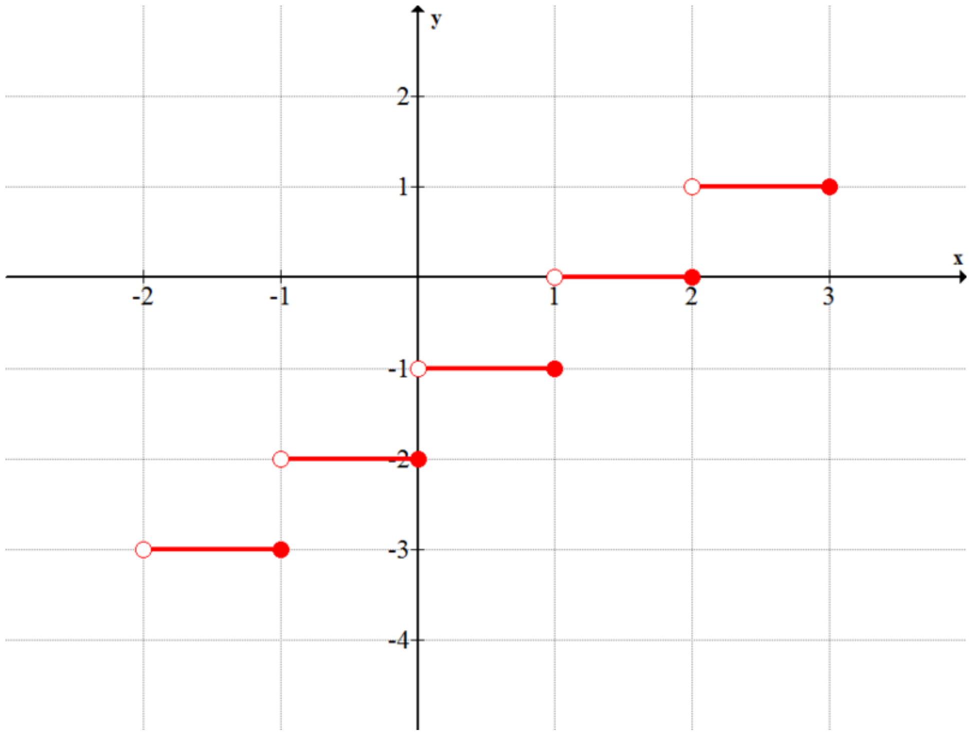
3. Is the relation discrete, continuous, or neither?
4. Find the domain and range for each of the three relations.

Use the graph below for #5 and #6.



5. Is the relation discrete, continuous, or neither?
6. Find the domain and range for the relation.

Use the graph below for #7 and #8.



7. Is the relation discrete, continuous, or neither?
8. Find the domain and range for the relation.

Examine the following pattern.

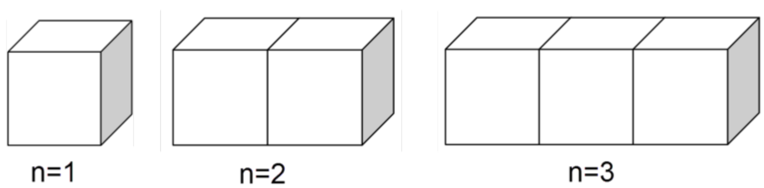


TABLE 4.5:

Number of Cubes (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**200**
Number of visible faces (f)	6	10	14						

9. Complete the table below the pattern.
10. Is the relation discrete, continuous, or neither?
11. Write a suitable domain and range for the pattern.

Examine the following pattern.

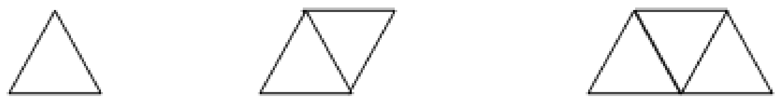


TABLE 4.6:

Number of triangles (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**100**
Number of toothpicks (t)									

12. Complete the table below the pattern.
13. Is the relation discrete, continuous, or neither?
14. Write a suitable domain and range for the pattern.

Examine the following pattern.

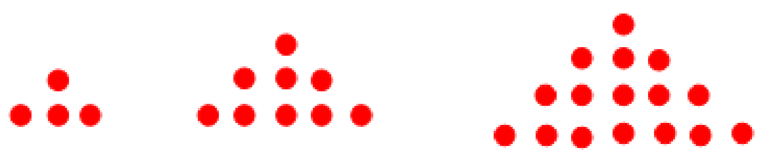


TABLE 4.7:

Pattern Number (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**100**
Number of dots (d)									

15. Complete the table below the pattern.
16. Is the relation discrete, continuous, or neither?
17. Write a suitable domain and range for the pattern.

4.3 Graphs of Functions based on Rules

Here you'll learn how to graph a function from a given rule.

What if you were given a function rule like $f(x) = \sqrt{2x^2+1}$. How could you graph that function? After completing this Concept, you'll be able to create a table of values to graph functions like this one in the coordinate plane.

Guidance

We can always make a graph from a function rule, by substituting values in for the variable and getting the corresponding output value.

Example A

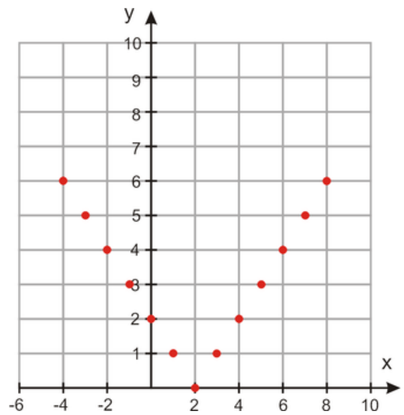
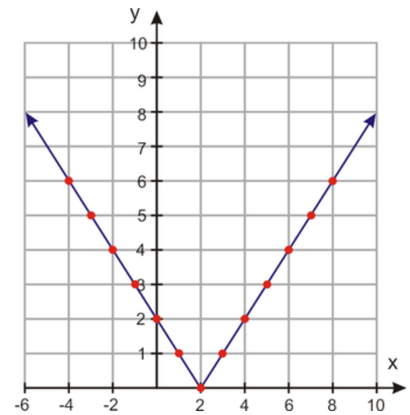
Graph the following function: $f(x) = |x-2|$

Solution

Make a table of values. Pick a variety of negative and positive values for x. Use the function rule to find the value of y for each value of x. Then, graph each of the coordinate points.

TABLE 4.8:

x	$y = f(x) = \mid x-2 \mid$
-4	$\mid -4-2 \mid = \mid -6 \mid = 6$
-3	$\mid -3-2 \mid = \mid -5 \mid = 5$
-2	$\mid -2-2 \mid = \mid -4 \mid = 4$
-1	$\mid -1-2 \mid = \mid -3 \mid = 3$
0	$\mid 0-2 \mid = \mid -2 \mid = 2$
1	$\mid 1-2 \mid = \mid -1 \mid = 1$
2	$\mid 2-2 \mid = \mid 0 \mid = 0$
3	$\mid 3-2 \mid = \mid 1 \mid = 1$
4	$\mid 4-2 \mid = \mid 2 \mid = 2$
5	$\mid 5-2 \mid = \mid 3 \mid = 3$
6	$\mid 6-2 \mid = \mid 4 \mid = 4$
7	$\mid 7-2 \mid = \mid 5 \mid = 5$
8	$\mid 8-2 \mid = \mid 6 \mid = 6$

It is wise to work with a lot of values when you begin graphing. As you learn about different types of functions, you will start to only need a few points in the table of values to create an accurate graph.

Example B

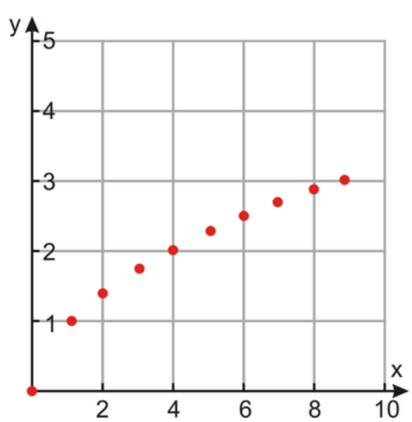
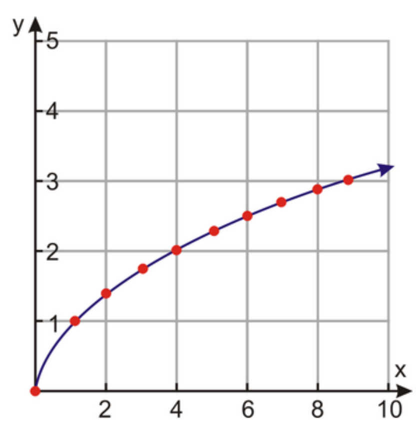
Graph the following function: $f(x) = \sqrt{x}$

Solution

Make a table of values. We know x can't be negative because we can't take the square root of a negative number. The domain is all positive real numbers, so we pick a variety of positive integer values for x. Use the function rule to find the value of y for each value of x.

TABLE 4.9:

x	$y = f(x) = \sqrt{x}$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	$\sqrt{4} = 2$
5	$\sqrt{5} \approx 2.24$
6	$\sqrt{6} \approx 2.45$
7	$\sqrt{7} \approx 2.65$
8	$\sqrt{8} \approx 2.83$
9	$\sqrt{9} = 3$

Note that the range is all positive real numbers.

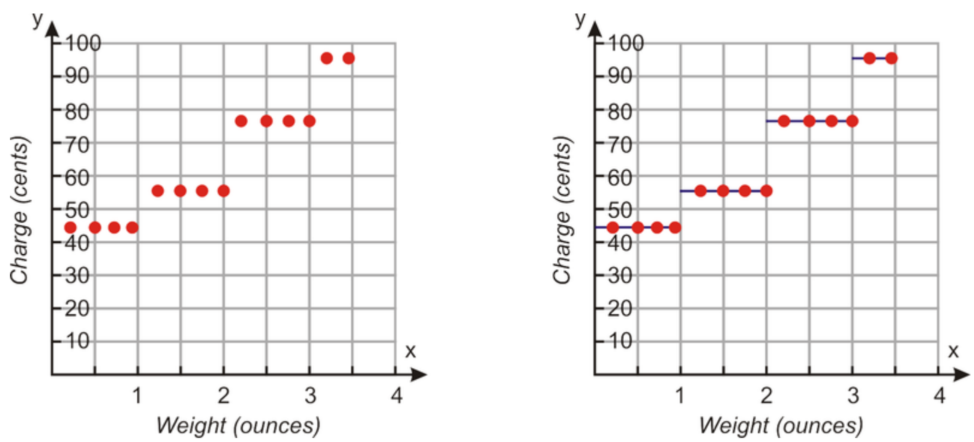
Example C

The post office charges 41 cents to send a letter that is one ounce or less and an extra 17 cents for each additional ounce or fraction of an ounce. This rate applies to letters up to 3.5 ounces.

Solution

Make a table of values. We can't use negative numbers for x because it doesn't make sense to have negative weight. We pick a variety of positive values for x, making sure to include some decimal values because prices can be decimals too. Then we use the function rule to find the value of y for each value of x.

x	0	0.2	0.5	0.8	1	1.2	1.5	1.8	2	2.2	2.5	2.8	3	3.2	3.5
y	0	41	41	41	41	58	58	58	58	75	75	75	75	92	92



Guided Practice

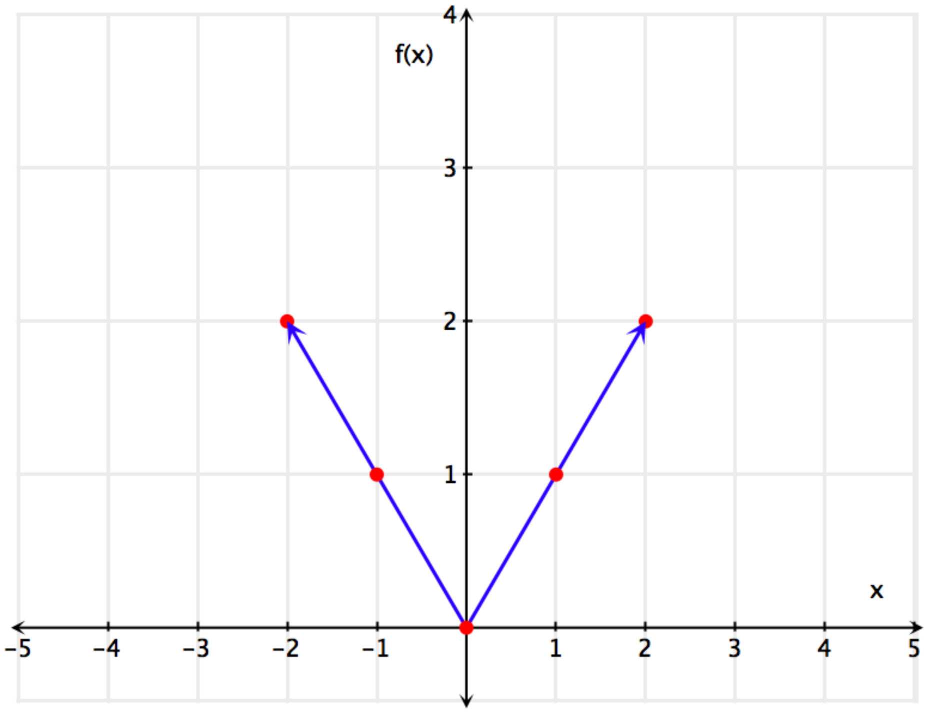
Graph the following function: $f(x) = \sqrt{x^2}$

Solution

Make a table of values. Even though x can't be negative inside the square root, because we are squaring x first, the domain is all real numbers. So we integer values for x which are on either side of zero. Use the function rule to find the value of y for each value of x.

TABLE 4.10:

x	$y = f(x) = \sqrt{x^2}$
-2	$\sqrt{(-2)^2} = 2$
-1	$\sqrt{(-1)^2} = 1$
0	$\sqrt{0^2} = 0$
1	$\sqrt{1^2} = 1$
2	$\sqrt{2^2} = 2$



Note that the range is all positive real numbers, and that this looks like an absolute value function.

Practice

Graph the following functions.

1. Vanson spends \$20 a month on his cat.
2. Brandon is a member of a movie club. He pays a \$50 annual membership and \$8 per movie.
3. $f(x) = (x-2)^2$
4. $f(x) = 3.2^x$
5. $f(t) = 27t - t^2$
6. $f(w) = \frac{w}{4} + 5$
7. $f(x) = t + 2t^2 + 3t^3$
8. $f(x) = (x-1)(x+3)$
9. $f(x) = \frac{x}{3} + \frac{x^2}{5}$
10. $f(x) = \sqrt{2x}$

4.4 Linear Interpolation and Extrapolation

Here you'll learn how to use linear interpolation to fill in gaps in data and linear extrapolation to estimate values outside a data set's range.

What if you were given a table of values that showed the average lifespan for Americans for each decade from 1950 to 2010? How could you use that data to estimate the average lifespan in 2020 or 2030? After completing this Concept, you'll be able to make predictions from linear models like this one.

Linear Interpolation

We use linear interpolation to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.

Example A

The following table shows the median ages of first marriage for men and women, as gathered by the U.S. Census Bureau.

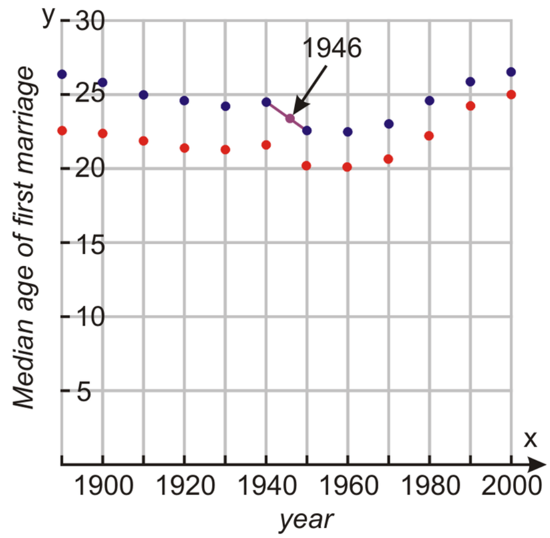
TABLE 4.11:

Year	Median age of males	Median age of females
1890	26.1	22.0
1900	25.9	21.9
1910	25.1	21.6
1920	24.6	21.2
1930	24.3	21.3
1940	24.3	21.5
1950	22.8	20.3
1960	22.8	20.3
1970	23.2	20.8
1980	24.7	22.0
1990	26.1	23.9
2000	26.8	25.1

Estimate the median age for the first marriage of a male in the year 1946.

Solution

We connect the two points on either side of 1946 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$m = \frac{22.8-24.3}{1950-1940} = \frac{-1.5}{10} = -0.15$$
$$y = -0.15x+b$$
$$24.3 = -0.15(1940)+b$$
$$b = 315.3$$

Our equation is $y = -0.15x+315.3$.

To estimate the median age of marriage of males in the year 1946, we plug $x = 1946$ into the equation we just found:

$y = -0.15(1946)+315.3 = 23.4$ **years old**

For non-linear data, linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation,** which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.

Linear Extrapolation

Linear extrapolation can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Example B

The winning times for the women's 100 meter race are given in the following table. Estimate the winning time in the year 2010. Is this a good estimate?

TABLE 4.12:

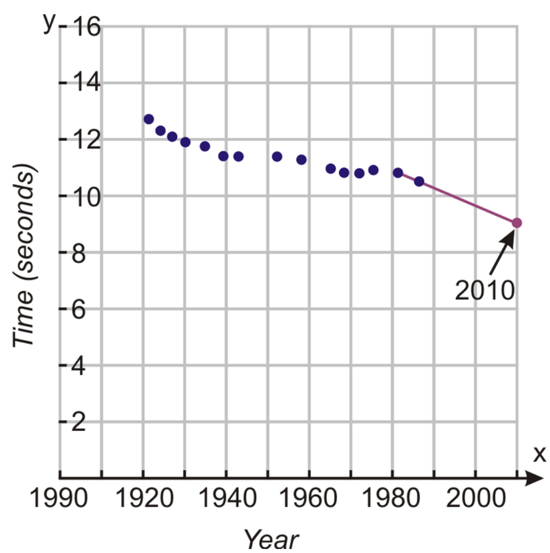
Winner	Country	Year	Time (seconds)
Mary Lines	UK	1922	12.8
Leni Schmidt	Germany	1925	12.4
Gerturd Glasitsch	Germany	1927	12.1
Tollien Schuurman	Netherlands	1930	12.0

TABLE 4.12: (continued)

Winner	Country	Year	Time (seconds)
Helen Stephens	USA	1935	11.8
Lulu Mae Hymes	USA	1939	11.5
Fanny Blankers-Koen	Netherlands	1943	11.5
Marjorie Jackson	Australia	1952	11.4
Vera Krepkina	Soviet Union	1958	11.3
Wyomia Tyus	USA	1964	11.2
Barbara Ferrell	USA	1968	11.1
Ellen Strophal	East Germany	1972	11.0
Inge Helten	West Germany	1976	11.0
Marlies Gohr	East Germany	1982	10.9
Florence Griffith Joyner	USA	1988	10.5

Solution

We start by making a scatter plot of the data; then we connect the last two points on the graph and find the equation of the line.



$$
\begin{aligned}
m &= \frac{10.5-10.9}{1988-1982} = \frac{-0.4}{6} = -0.067 \\
y &= -0.067x + b \\
10.5 &= -0.067(1988) + b \\
b &= 143.7
\end{aligned}
$$

Our equation is $y = -0.067x + 143.7$.

The winning time in year 2010 is estimated to be:

$y = -0.067(2010) + 143.7 = 9.03$ **seconds.**

Unfortunately, this estimate actually isn't very accurate. This example demonstrates the weakness of linear extrapolation; it uses only a couple of points, instead of using *all* the points like the best fit line method, so it doesn't give as accurate results when the data points follow a linear pattern. In this particular example, the last data point clearly doesn't fit in with the general trend of the data, so the slope of the extrapolation line is much steeper than it would be if we'd used a line of best fit. (As a historical note, the last data point corresponds to the winning time for Florence Griffith Joyner in 1988. After her race she was accused of using performance-enhancing drugs, but this fact was never proven. In addition, there was a question about the accuracy of the timing: some officials said that tail-wind was not accounted for in this race, even though all the other races of the day were affected by a strong wind.)

Here's an example of a problem where linear extrapolation does work better than the line of best fit method.

Example C

A cylinder is filled with water to a height of 73 centimeters. The water is drained through a hole in the bottom of the cylinder and measurements are taken at 2 second intervals. The following table shows the height of the water level in the cylinder at different times.

TABLE 4.13:

Time (seconds)	Water level (cm)
0.0	73
2.0	63.9
4.0	55.5
6.0	47.2
8.0	40.0
10.0	33.4
12.0	27.4
14.0	21.9
16.0	17.1
18.0	12.9
20.0	9.4
22.0	6.3
24.0	3.9
26.0	2.0
28.0	0.7
30.0	0.1

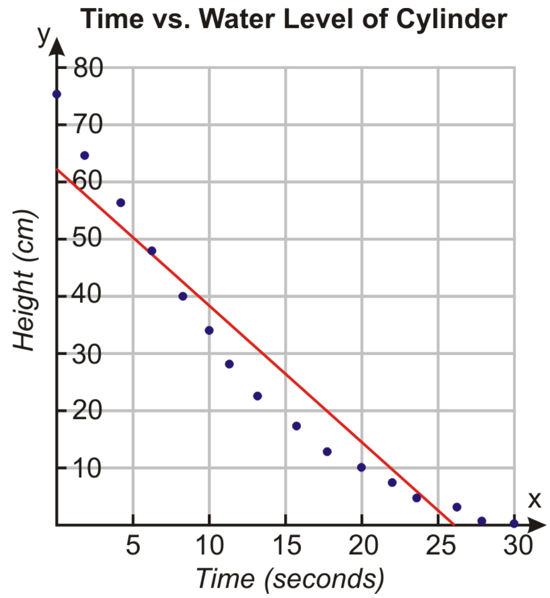
a) *Find the water level at time 15 seconds.*

b) *Find the water level at time 27 seconds*

c) *What would be the original height of the water in the cylinder if the water takes 5 extra seconds to drain? (Find the height at time of –5 seconds.)*

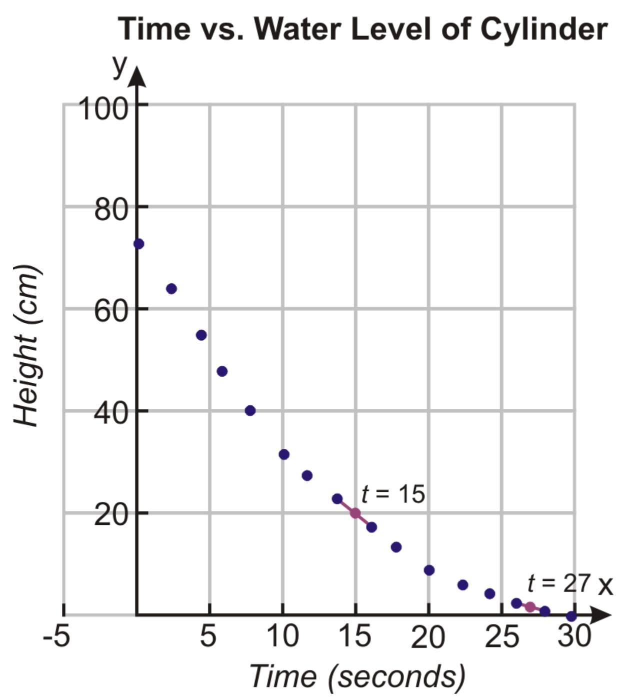
Solution

Here's what the line of best fit would look like for this data set:



Notice that the data points don't really make a line, and so the line of best fit still isn't a terribly good fit. Just a glance tells us that we'd estimate the water level at 15 seconds to be about 27 cm, which is *more* than the water level at 14 seconds. That's clearly not possible! Similarly, at 27 seconds we'd estimate the water to have all drained out, which it clearly hasn't yet.

So let's see what happens if we use linear extrapolation and interpolation instead. First, here are the lines we'd use to interpolate between 14 and 16 seconds, and between 26 and 28 seconds.



a) The slope of the line between points (14, 21.9) and (16, 17.1) is $m = \frac{17.1-21.9}{16-14} = \frac{-4.8}{2} = -2.4$. So $y = -2.4x + b \Rightarrow 21.9 = -2.4(14) + b \Rightarrow b = 55.5$, and the equation is $y = -2.4x + 55.5$.

Plugging in $x = 15$ gives us $y = -2.4(15) + 55.5 = 19.5\ cm$.

b) The slope of the line between points (26, 2) and (28, 0.7) is $m = \frac{0.7-2}{28-26} = \frac{-1.3}{2} = -.65$, so $y = -.65x + b \Rightarrow 2 = -.65(26) + b \Rightarrow b = 18.9$, and the equation is $y = -.65x + 18.9$.

Plugging in $x = 27$, we get $y = -.65(27) + 18.9 = 1.35\ cm$.

c) Finally, we can use extrapolation to estimate the height of the water at -5 seconds. The slope of the line between points (0, 73) and (2, 63.9) is $m = \frac{63.9-73}{2-0} = \frac{-9.1}{2} = -4.55$, so the equation of the line is $y = -4.55x + 73$.

Plugging in $x = -5$ gives us $y = -4.55(-5) + 73 = 95.75\ cm$.

Vocabulary

- The **line of best fit** is a good method if the relationship between the dependent and the independent variables is linear. In this section you will learn other methods that are useful even when the relationship isn't linear.
- We use **linear interpolation** to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.
- **For non-linear data,** linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation,** which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.
- **Linear extrapolation** can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Guided Practice

The Center for Disease Control collects information about the health of the American people and behaviors that might lead to bad health. The following table shows the percent of women who smoke during pregnancy.

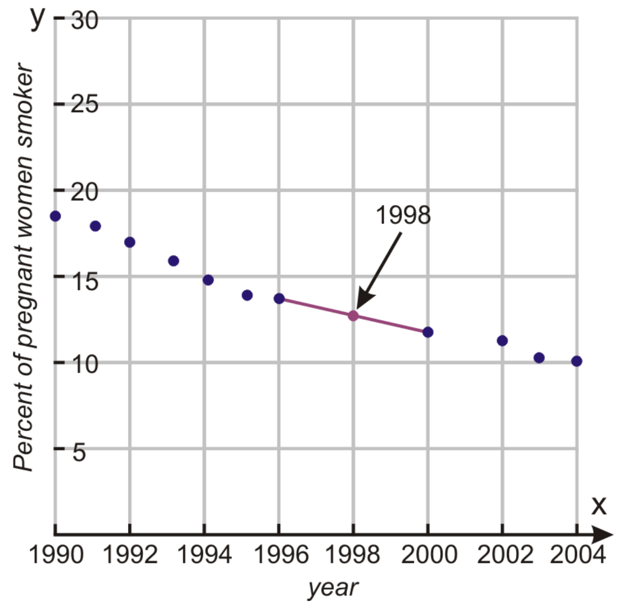
TABLE 4.14:

Year	Percent of pregnant women smokers
1990	18.4
1991	17.7
1992	16.9
1993	15.8
1994	14.6
1995	13.9
1996	13.6
2000	12.2
2002	11.4
2003	10.4
2004	10.2

Estimate the percentage of pregnant women that were smoking in the year 1998.

Solution

We connect the two points on either side of 1998 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$\begin{aligned} m &= \frac{12.2-13.6}{2000-1996} = \frac{-1.4}{4} = -0.35 \\ y &= -0.35x+b \\ 12.2 &= -0.35(2000)+b \\ b &= 712.2 \end{aligned}$$

Our equation is $y = -0.35x + 712.2$.

To estimate the percentage of pregnant women who smoked in the year 1998, we plug $x = 1998$ into the equation we just found:

$$y = -0.35(1998) + 712.2 = 12.9\%$$

Practice

1. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1946. Fit a line, by hand, to the data before 1970.
2. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1984. Fit a line, by hand, to the data from 1970 on in order to estimate this accurately.
3. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for males in 1995. Use linear interpolation between the 1990 and 2000 data points.
4. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 1997. Use linear interpolation between the 1996 and 2000 data points.
5. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 2006. Use linear extrapolation with the final two data points.
6. Use the data from Example 3 (*Winning times*) to estimate the winning time for the female 100-meter race in 1920. Use linear extrapolation because the first two or three data points have a different slope than the rest of the data.
7. The table below shows the highest temperature vs. the hours of daylight for the 15^{th} day of each month in the year 2006 in San Diego, California.

TABLE 4.15:

Hours of daylight	High temperature (F)
10.25	60
11.0	62
12	62
13	66
13.8	68
14.3	73
14	86
13.4	75
12.4	71
11.4	66
10.5	73
10	61

(a) What would be a better way to organize this table if you want to make the relationship between daylight hours and temperature easier to see?

(b) Estimate the high temperature for a day with 13.2 hours of daylight using linear interpolation.

(c) Estimate the high temperature for a day with 9 hours of daylight using linear extrapolation. Is the prediction accurate?

(d) Estimate the high temperature for a day with 9 hours of daylight using a line of best fit.

The table below lists expected life expectancies based on year of birth (US Census Bureau). Use it to answer questions 8-15.

TABLE 4.16:

Birth year	Life expectancy in years
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77

1. Make a scatter plot of the data.
2. Use a line of best fit to estimate the life expectancy of a person born in 1955.
3. Use linear interpolation to estimate the life expectancy of a person born in 1955.
4. Use a line of best fit to estimate the life expectancy of a person born in 1976.
5. Use linear interpolation to estimate the life expectancy of a person born in 1976.
6. Use a line of best fit to estimate the life expectancy of a person born in 2012.
7. Use linear extrapolation to estimate the life expectancy of a person born in 2012.
8. Which method gives better estimates for this data set? Why?

The table below lists the high temperature for the fist day of the month for the year 2006 in San Diego, California (Weather Underground). Use it to answer questions 16-21.

TABLE 4.17:

Month number	Temperature (F)
1	63
2	66
3	61
4	64
5	71
6	78
7	88
8	78
9	81
10	75
11	68
12	69

1. Draw a scatter plot of the data.
2. Use a line of best fit to estimate the temperature in the middle of the 4^{th} month (month 4.5).
3. Use linear interpolation to estimate the temperature in the middle of the 4^{th} month (month 4.5).
4. Use a line of best fit to estimate the temperature for month 13 (January 2007).
5. Use linear extrapolation to estimate the temperature for month 13 (January 2007).
6. Which method gives better estimates for this data set? Why?
7. Name a real-world situation where you might want to make predictions based on available data. Would linear extrapolation/interpolation or the best fit method be better to use in that situation? Why?

4.5 Inequality Expressions

Here you'll learn how to write and graph inequalities in one variable on a number line.

Guidance

Dita has a budget of $350 to spend on a rental car for an upcoming trip, but she wants to spend as little of that money as possible. If the trip will last five days, what range of daily rental rates should she be willing to consider?

Like equations, inequalities show a relationship between two expressions. We solve and graph inequalities in a similar way to equations—but when we solve an inequality, the answer is usually a set of values instead of just one value.

When writing inequalities we use the following symbols:

$>$is greater than

$\geq$ is greater than or equal to

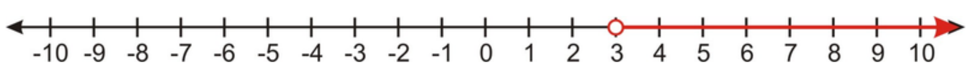
$<$is less than

$\leq$ is less than or equal to

Write and Graph Inequalities in One Variable on a Number Line

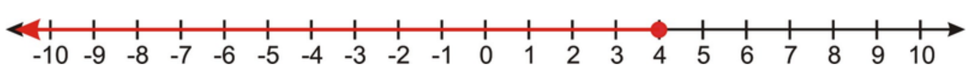
Let's start with the simple inequality $x > 3$.

We read this inequality as "x is greater than 3." The solution is the set of all real numbers that are greater than three. We often represent the solution set of an inequality with a number line graph.



Consider another simple inequality: $x \leq 4$.

We read this inequality as "x is less than or equal to 4." The solution is the set of all real numbers that are equal to four or less than four. We can graph this solution set on the number line.



Notice that we use an empty circle for the endpoint of a strict inequality (like $x > 3$), and a filled circle for one where the equals sign is included (like $x \leq 4$).

Example A

Graph the following inequalities on the number line.

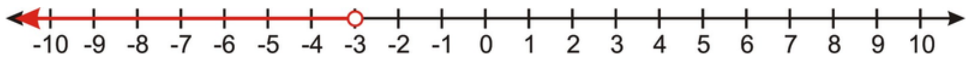
a) $x < -3$

b) $x \geq 6$

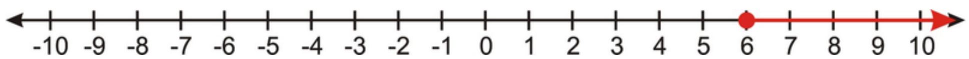
c) $x > 0$

Solution

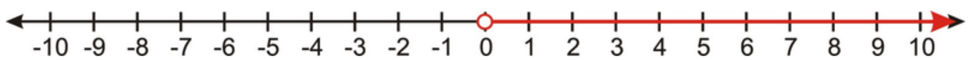
a) The inequality $x < -3$ represents all numbers that are less than -3. The number -3 is not included in the solution, so it is represented by an open circle on the graph.



b) The inequality $x \geq 6$ represents all numbers that are greater than or equal to 6. The number 6 is included in the solution, so it is represented by a closed circle on the graph.



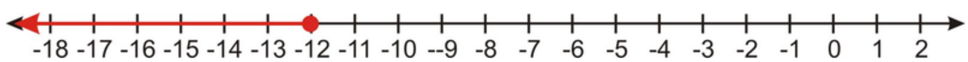
c) The inequality $x > 0$ represents all numbers that are greater than 0. The number 0 is not included in the solution, so it is represented by an open circle on the graph.



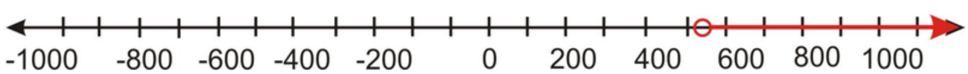
Example B

Write the inequality that is represented by each graph.

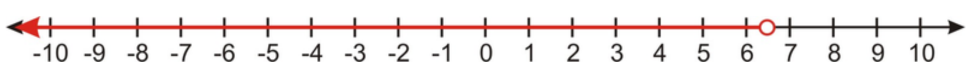
a)



b)



c)



Solution

a) $x \leq -12$

b) $x > 540$

c) $x < 6.5$

Inequalities appear everywhere in real life. Here are some simple examples of real-world applications.

Example C

Write each statement as an inequality and graph it on the number line.

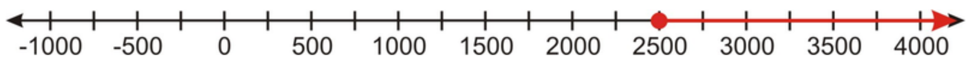
a) You must maintain a balance of at least $2500 in your checking account to get free checking.

b) You must be at least 48 inches tall to ride the “Thunderbolt” Rollercoaster.

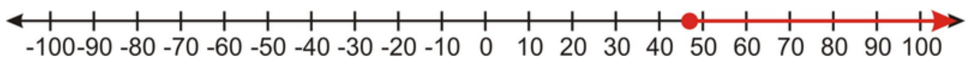
c) You must be younger than 3 years old to get free admission at the San Diego Zoo.

Solution

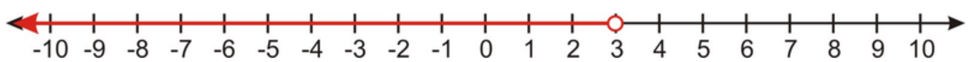
a) The words "at least" imply that the value of $2500 is included in the solution set, so the inequality is written as $x \geq 2500$.



b) The words "at least" imply that the value of 48 inches is included in the solution set, so the inequality is written as $x \geq 48$.



c) The inequality is written as $x < 3$.



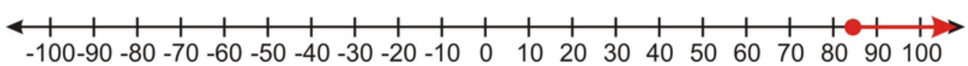
Vocabulary

- The answer to an **inequality** is usually an **interval of values**.

Guided Practice

1. *Graph the inequality* $x \leq 8$ *on the number line.*

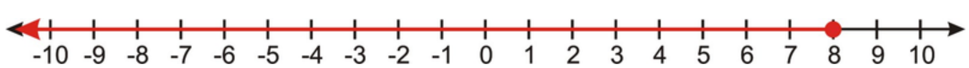
2. *Write the inequality that is represented by the graph below.*



3. *Write the statement, "the speed limit on the interstate is 65 miles per hour or less" as an inequality .*

Solution

1. The inequality $x \leq 8$ represents all numbers that are less than or equal to 8. The number 8 is included in the solution, so it is represented by a closed circle on the graph.

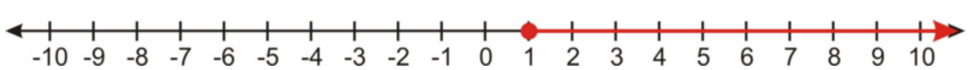


2. $x \geq 85$

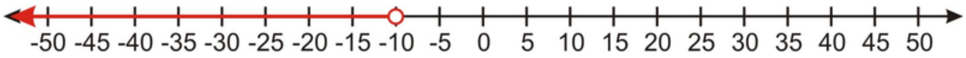
3. Speed limit means the highest allowable speed, so the inequality is written as $x \leq 65$.

Practice

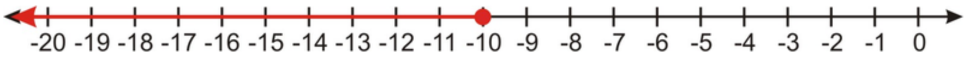
1. Write the inequality represented by the graph.



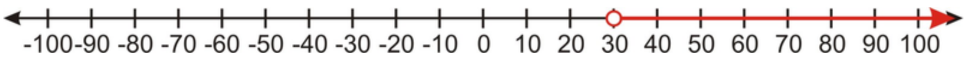
2. Write the inequality represented by the graph.



3. Write the inequality represented by the graph.



4. Write the inequality represented by the graph.



Graph each inequality on the number line.

5. $x < -35$
6. $x > -17$
7. $x \geq 20$
8. $x \leq 3$
9. $x \geq -5$
10. $x > 20$

4.6 Compound Inequalities

Here you will solve two inequalities that have been joined together by the words "and" and "or."

Guidance

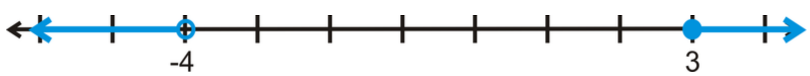
Compound inequalities are inequalities that have been joined by the words "and" or "or." For example:

-2 <x <5 Read, "x is greater than -2 *and* less than 5."

$x \geq 3$ or $x < -4$ Read, "x is greater than or equal to 3 *or* less than -4."

Notice that both of these inequalities have two inequality signs. So, it is like solving or graphing two inequalities at the same time. When graphing, look at the inequality to help you. The first compound inequality above, $-2 < x \leq 5$, has the x in between -2 and 5, so the shading will also be between the two numbers.

And, with the "or" statement, the shading will go in opposite directions.



Example A

Write the inequality statement given by the graph below.

Solution: Because the shading goes in opposite directions, we know this is an "or" statement. Therefore, the statement is $x < -2$ or $x > 1$.

Example B

Solve and graph $-3 < 2x + 5 \leq 11$.

Solution: This is like solving two inequalities at the same time. You can split the statement apart to have two inequalities, $-3 < 2x + 5$ and $2x + 5 \leq 11$ and solve. You can also leave the compound inequality whole to solve.

$$\begin{aligned} &-3 < 2x + \cancel{5} \leq 11 \\ &\underline{-5 \qquad -\cancel{5} \quad -5} \\ &\frac{-8}{2} < \frac{\cancel{2}x}{\cancel{2}} \leq \frac{6}{2} \\ &-4 < x \leq 3 \end{aligned}$$

Test a solution, $x = 0$:

$$\begin{aligned} -3 < 2(0) + 5 \leq 11 \\ -3 < 5 \leq 11 \end{aligned}$$

Here is the graph:

Example C

Solve and graph $-32 > -5x + 3$ or $x - 4 \leq 2$.

Solution: When solving an "or" inequality, solve the two inequalities separately, but show the solution on the same number line.

$$\begin{aligned} -32 > -5x + 3 \quad & or \quad x - 4 \leq 2 \\ \underline{-3 \qquad\quad -3} \quad & \quad \underline{+4 \quad +4} \\ \frac{-35}{-5} > \frac{-5x}{-5} \quad & \qquad x \leq 6 \\ 7 < x \quad & \end{aligned}$$

Notice that in the first inequality, we had to flip the inequality sign because we divided by -5. Also, it is a little more complicated to test a solution for these types of inequalities. You still test one point, but it will only work for one of the inequalities. Let's test $x = 10$. First inequality: $-32 > -5(10) + 3 \rightarrow -32 > -47$. Second inequality: $10 - 4 \leq 2 \rightarrow 5 \not\leq 2$. Because $x = 10$ works for the first inequality, is a solution. Here is the graph.

Intro Problem Revisit Writing the grading as an expression, we get $0.4(84) + 0.6x$ where x is the final exam score. Madison wants to get an A, so we will have a compound inequality that ranges between 90 and 100.

$$\begin{aligned} 90 \leq 33.6 + 0.6x \leq 100 \\ 56.4 \leq 0.6x \leq 66.4 \\ 94 \leq x \leq 110.67 \end{aligned}$$

Unless Mr. Garcia offers extra credit, Madison can't score higher than 100. So, she has to score at least 94 or more, up to 100, to get an A.

Guided Practice

1. Graph $-7 \leq x \leq -1$ on a number line.

Solve the following compound inequalities and graph.

2. $5 \leq -\frac{2}{3}x + 1 \leq 15$

3. $\frac{x}{4} - 7 > 5$ or $\frac{8}{5}x + 2 \leq 18$

Answers

1. This is an "and" inequality, so the shading will be between the two numbers.

2. Solve this just like Example B.

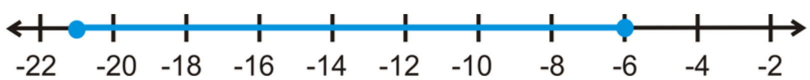
$$\begin{aligned} 5 \le -\frac{2}{3}x + \cancel{1} \le 15 \\ \underline{-1 \qquad\quad -\cancel{1} \quad -1} \\ 4 \le -\frac{2}{3}x \le 14 \\ -\frac{3}{2}\left(4 \le -\frac{2}{3}x \le 14\right) \\ -6 \ge x \ge -21 \end{aligned}$$

Test a solution, $x = -10$:

$$\begin{aligned} 5 \le -\frac{2}{3}(-12) + 1 \le 15 \\ 5 \le 9 \le 15 \end{aligned}$$

This solution can also be written $-21 \le x \le -6$.

The graph is:



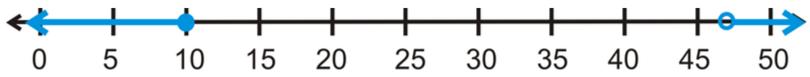
3. This is an "or" compound inequality. Solve the two inequalities separately.

$$\begin{aligned} \frac{x}{4} - \cancel{7} > 5 \qquad &or \qquad \frac{8}{5}x + \cancel{2} \le 18 \\ \underline{+\cancel{7} \ +7} \qquad\quad & \qquad\quad \underline{-\cancel{2} \ -2} \\ \cancel{4} \cdot \frac{x}{4} > 12 \cdot 4 \ &or \ \cancel{\frac{5}{8}} \cdot \cancel{\frac{8}{5}}x \le 16 \cdot \frac{5}{8} \\ x > 48 \qquad &or \qquad x \le 10 \end{aligned}$$

Test a solution, $x = 0$:

$$\begin{aligned} \frac{0}{4} - 7 > 5 \ \ or \ \ \frac{8}{5}(0) + 2 \le 18 \\ -7 \not> 5 \ \ or \qquad 2 \le 18 \end{aligned}$$

Notice that $x = 0$ is a solution for the second inequality, which makes it a solution for the entire compound inequality. Here is the graph:



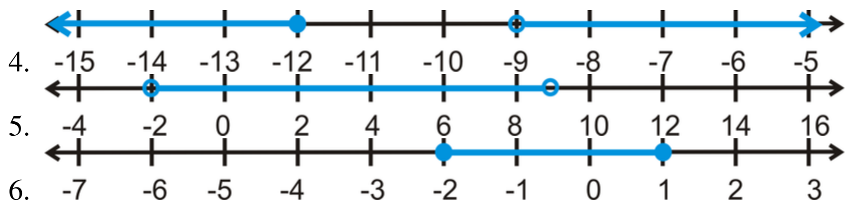
On problems 2 and 3 we changed the scale of the number line to accommodate the solution.

Practice

Graph the following compound inequalities. Use an appropriate scale.

1. $-1 < x < 8$
2. $x > 5$ or $x \leq 3$
3. $-4 \leq x \leq 0$

Write the compound inequality that best fits each graph below.



Solve each compound inequality and graph the solution.

7. $-11 < x - 9 \leq 2$
8. $8 \leq 3 - 5x < 28$
9. $2x - 7 > -13$ or $\frac{1}{3}x + 5 \leq 1$
10. $0 < \frac{x}{5} < 4$
11. $-4x + 9 < 35$ or $3x - 7 \leq -16$
12. $\frac{3}{4}x + 7 \geq -29$ or $16 - x > 2$
13. $3 \leq 6x - 15 < 51$
14. $-20 < -\frac{3}{2}x + 1 < 16$
15. **Challenge** Write a compound inequality whose solutions are all real numbers. Show why this is true.

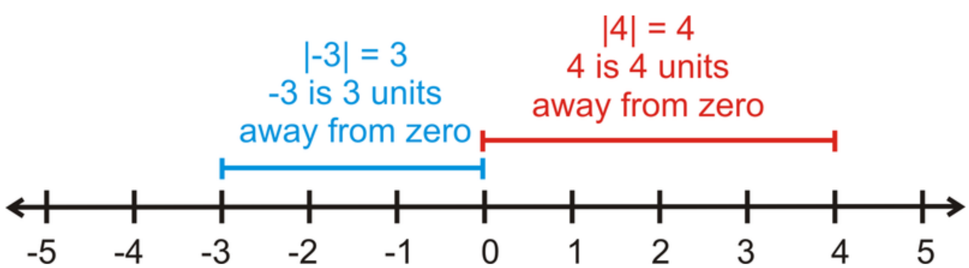
4.7 Solving Absolute Value Equations

Here you'll learn to solve absolute value equations.

To determine the height of skeletal remains, archaeologists use the equation $H = 2.26f + 66.4$, where H is the height in centimeters and f is the length of the skeleton's femur (also in cm). The equation has a margin of error of $\pm 3.42cm$. Dr. Jordan found a skeletal femur that is 46.8 cm. Determine the greatest height and the least height of this person.

Guidance

Absolute value is the distance a number is from zero. Because distance is always positive, the absolute value will always be positive. Absolute value is denoted with two vertical lines around a number, $|x|$.



$|5| = 5$ $|-9| = 9$ $|0| = 0$ $|-1| = 1$

When solving an absolute value equation, the value of x could be two different possibilities; whatever makes the absolute value positive OR whatever makes it negative. Therefore, there will always be TWO answers for an absolute value equation.

If $|x| = 1$, then x can be 1 or -1 because $|1| = 1$ and $|-1| = 1$.

If $|x| = 15$, then x can be 15 or -15 because $|15| = 15$ and $|-15| = 15$.

From these statements we can conclude:

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Example A

Determine if $x = -12$ is a solution to $|2x - 5| = 29$.

Solution: Plug in -12 for x to see if it works.

$$\begin{aligned} |2(-12) - 5| &= 29 \\ |-24 - 5| &= 29 \\ |-29| &= 29 \end{aligned}$$

-12 is a solution to this absolute value equation.

Example B

Solve $|x+4|=11$.

Solution: There are going to be two answers for this equation. $x+4$ can equal 11 or -11.

$$
\begin{array}{ccc}
 & |x+4|=11 & \\
 & \swarrow \searrow & \\
x+4=11 & & x+4=-11 \\
 & or & \\
x=7 & & x=-15
\end{array}
$$

Test the solutions:

$$
\begin{array}{cc}
|7+4|=11 & |-15+4|=11 \\
|11|=11 & |-11|=11
\end{array}
$$

Example C

Solve $\left|\frac{2}{3}x-5\right|=17$.

Solution: Here, what is inside the absolute value can be equal to 17 or -17.

$$
\begin{array}{ccc}
 & \left|\frac{2}{3}x-5\right|=17 & \\
 & \swarrow \searrow & \\
\frac{2}{3}x-5=17 & & \frac{2}{3}x-5=-17 \\
\frac{2}{3}x=22 & or & \frac{2}{3}x=-12 \\
x=22\cdot\frac{3}{2} & & x=-12\cdot\frac{3}{2} \\
x=33 & & x=-18
\end{array}
$$

Test the solutions:

$$
\begin{array}{cc}
\left|\frac{2}{3}(33)-5\right|=17 & \left|\frac{2}{3}(-18)-5\right|=17 \\
|22-5|=17 & |-12-5|=17 \\
|17|=17 & |-17|=17
\end{array}
$$

Intro Problem Revisit First, we need to find the height of the skeleton using the equation $H=2.26f+66.4$, where $f=46.8$.

$$H = 2.26(46.8) + 66.4$$
$$H = 172.168cm$$

Now, let's use an absolute value equation to determine the margin of error, and thus the greatest and least heights.

$$|x - 172.168| = 3.42$$
$$\swarrow \searrow$$
$$x - 172.168 = 3.42 \quad x - 172.168 = -3.42$$
$$or$$
$$x = 175.588 \qquad x = 168.748$$

So the person could have been a maximum of 175.588 cm or a minimum of 168.748 cm. In inches, this would be 69.13 and 66.44 inches, respectively.

Guided Practice

1. Is $x = -5$ a solution to $|3x + 22| = 6$?

Solve the following absolute value equations.

2. $|6x - 11| + 2 = 41$

3. $\left|\frac{1}{2}x + 3\right| = 9$

Answers

1. Plug in -5 for x to see if it works.

$$|3(-5) + 22| = 6$$
$$|-15 + 22| = 6$$
$$|-7| \neq 6$$

-5 is not a solution because $|-7| = 7$, not 6.

2. Find the two solutions. Because there is a 2 being added to the left-side of the equation, we first need to subtract it from both sides so the absolute value is by itself.

$$|6x - 11| + 2 = 41$$
$$|6x - 11| = 39$$
$$\swarrow \searrow$$
$$6x - 11 = 39 \quad 6x - 11 = -39$$
$$6x = 50 \qquad 6x = -28$$
$$x = \frac{50}{6} \quad or \quad x = -\frac{28}{6}$$
$$= \frac{25}{3} \; or \; 8\frac{1}{3} \qquad = -\frac{14}{3} \; or \; -4\frac{2}{3}$$

Check both solutions. It is easier to check solutions when they are improper fractions.

$$\left|6\left(\frac{25}{3}\right)-11\right|=39 \qquad \left|6\left(-\frac{14}{3}\right)-11\right|=39$$
$$|50-11|=39 \quad \textit{and} \quad |-28-11|=39$$
$$|39|=39 \qquad |-39|=39$$

3. What is inside the absolute value is equal to 9 or -9.

$$\left|\frac{1}{2}x+3\right|=9$$
$$\swarrow \searrow$$
$$\frac{1}{2}x+3=9 \qquad \frac{1}{2}x+3=-9$$
$$\frac{1}{2}x=6 \quad or \quad \frac{1}{2}x=-12$$
$$x=12 \qquad x=-24$$

Test solutions:

$$\left|\frac{1}{2}(12)+3\right|=9 \qquad \left|\frac{1}{2}(-24)+3\right|=9$$
$$|6+3|=9 \qquad |-12+3|=9$$
$$|9|=9 \qquad |-9|=9$$

Vocabulary

Absolute Value
The positive distance from zero a given number is.

Practice

Determine if the following numbers are solutions to the given absolute value equations.

1. $|x-7|=16; 9$
2. $|\frac{1}{4}x+1|=4; -8$
3. $|5x-2|=7; -1$

Solve the following absolute value equations.

4. $|x+3|=8$
5. $|2x|=9$
6. $|2x+15|=3$
7. $\left|\frac{1}{3}x-5\right|=2$
8. $\left|\frac{x}{6}+4\right|=5$

9. $|7x-12|=23$
10. $\left|\frac{3}{5}x+2\right|=11$
11. $|4x-15|+1=18$
12. $|-3x+20|=35$
13. $|12x-18|=0$
14. What happened in #13? Why do you think that is?
15. **Challenge** When would an absolute value equation have no solution? Give an example.

4.8 Solving Absolute Value Inequalities

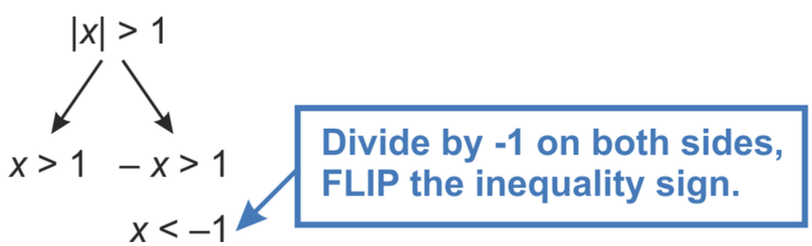
Here you'll learn how to solve absolute value inequalities.

The tolerance for the weight of a volleyball is 2.6 grams. If the average volleyball weighs 260 grams, what is the range of weights for a volleyball?

Guidance

Like absolute value equations, absolute value inequalities also will have two answers. However, they will have a range of answers, just like compound inequalities.

$|x| > 1$ This inequality will have two answers, when x is 1 and when $-x$ is 1. But, what about the inequality sign? The two possibilities would be:



Notice in the second inequality, we did not write $x > -1$. This is because what is inside the absolute value sign can be positive or negative. Therefore, if x is negative, then $-x > 1$. It is a very important difference between the two inequalities. Therefore, for the first solution, we leave the inequality sign the same and for the second solution we need to change the sign of the answer AND flip the inequality sign.

Example A

Solve $|x+2| \leq 10$.

Solution: There will be two solutions, one with the answer and sign unchanged and the other with the inequality sign flipped and the answer with the opposite sign.

$$|x+2| \leq 10$$

$$\swarrow \searrow$$

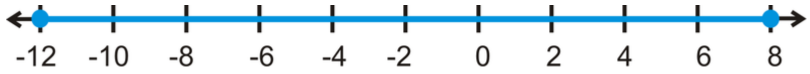
$$x+2 \leq 10 \qquad x+2 \geq -10$$

$$x \leq 8 \qquad x \geq -12$$

Test a solution, $x = 0$:

$$|0+2| \leq 10$$
$$|2| \leq 10$$

When graphing this inequality, we have



Notice that this particular absolute value inequality has a solution that is an "and" inequality because the solution is between two numbers.

If $|ax+b| < c$ where $a > 0$ and $c > 0$, then $-c < ax+b < c$.

If $|ax+b| \leq c$ where $a > 0$ and $c > 0$, then $-c \leq ax+b \leq c$.

If $|ax+b| > c$ where $a > 0$ and $c > 0$, then $ax+b < -c$ or $ax+b > c$.

If $|ax+b| \geq c$ where $a > 0$ and $c > 0$, then $ax+b \leq -c$ or $ax+b \geq c$.

If $a < 0$, we will have to divide by a negative and have to flip the inequality sign. This would change the end result. If you are ever confused by the rules above, always test one or two solutions and graph it.

Example B

Solve and graph $|4x-3| > 9$.

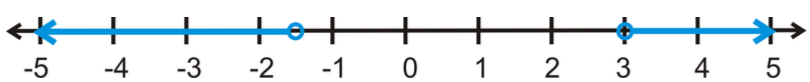
Solution: Break apart the absolute value inequality to find the two solutions.

$$|4x-3| > 9$$
$$\swarrow \searrow$$
$$\begin{array}{ll} 4x-3 > 9 & 4x-3 < -9 \\ 4x > 12 & 4x < -6 \\ x > 3 & x < -\frac{3}{2} \end{array}$$

Test a solution, $x = 5$:

$$\begin{aligned} |4(5)-3| &> 9 \\ |20-3| &> 9 \\ 17 &> 9 \end{aligned}$$

The graph is:



Example C

Solve $|-2x+5| < 11$.

Solution: In this example, the rules above do not apply because $a < 0$. At first glance, this should become an "and" inequality. But, because we will have to divide by a negative number, a, the answer will be in the form of an "or" compound inequality. We can still solve it the same way we have solved the other examples.

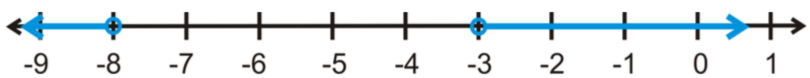
$$|-2x+5| < 11$$

$$\swarrow \searrow$$

$$\begin{aligned} -2x+5 &< 11 & -2x+5 &> -11 \\ -2x &< 6 & -2x &> -16 \\ x &> -3 & x &< -8 \end{aligned}$$

The solution is less than -8 or greater than -3.

The graph is:



When $a < 0$ for an absolute value inequality, it switches the results of the rules listed above.

Intro Problem Revisit Set up an absolute value inequality. w is the range of weights of the volleyball.

$$|w-260| \leq 2.6$$

$$\swarrow \searrow$$

$$\begin{aligned} w-260 &\leq 2.6 & w-260 &\geq -2.6 \\ w &\leq 262.6 & w &\geq 257.4 \end{aligned}$$

So, the range of the weight of a volleyball is $257.4 \leq w \leq 262.6$ grams.

Guided Practice

1. Is $x = -4$ a solution to $|15-2x| > 9$?

2. Solve and graph $\left|\frac{2}{3}x+5\right| \leq 17$.

Answers

1. Plug in -4 for x to see if it works.

$$\begin{aligned} |15-2(-4)| &> 9 \\ |15+8| &> 9 \\ |23| &> 9 \\ 23 &> 9 \end{aligned}$$

Yes, -4 works, so it is a solution to this absolute value inequality.

2. Split apart the inequality to find the two answers.

$$\left|\frac{2}{3}x+5\right| \leq 17$$

$$\swarrow \searrow$$

$$\left|\frac{2}{3}x+5\right| \leq 17 \qquad \frac{2}{3}x+5 \geq -17$$

$$\frac{2}{3}x \leq 12 \qquad \frac{2}{3}x \geq -22$$

$$x \leq 12 \cdot \frac{3}{2} \qquad x \geq -22 \cdot \frac{3}{2}$$

$$x \leq 18 \qquad x \geq -33$$

Test a solution, $x = 0:$

$$\left|\frac{2}{3}(0)+5\right| \leq 17$$

$$|5| \leq 17$$

$$5 \leq 17$$

Practice

Determine if the following numbers are solutions to the given absolute value inequalities.

1. $|x-9| > 4; 10$
2. $\left|\frac{1}{2}x-5\right| \leq 1; 8$
3. $|5x+14| \geq 29; -8$

Solve and graph the following absolute value inequalities.

4. $|x+6| > 12$
5. $|9-x| \leq 16$
6. $|2x-7| \geq 3$
7. $|8x-5| < 27$
8. $\left|\frac{5}{6}x+1\right| > 6$
9. $|18-4x| \leq 2$
10. $\left|\frac{3}{4}x-8\right| > 13$
11. $|6-7x| \leq 34$
12. $|19+3x| \geq 46$

Solve the following absolute value inequalities. a is greater than zero.

13. $|x-a| > a$
14. $|x+a| \leq a$
15. $|a-x| \leq a$

4.9 Linear Inequalities in Two Variables

Here you'll learn how to graph linear inequalities in two variables of the form $y > mx+b$ or $y < mx+b$. You'll also solve real-world problems involving such inequalities.

Guidance

The general procedure for graphing inequalities in two variables is as follows:

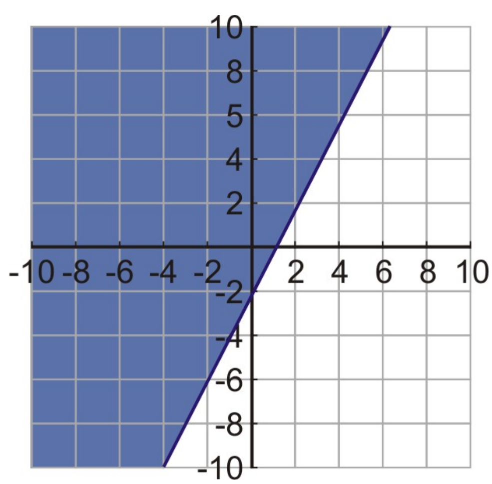
1. Re-write the inequality in slope-intercept form: $y = mx+b$. Writing the inequality in this form lets you know the direction of the inequality.
2. Graph the line of the equation $y = mx+b$ using your favorite method (plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
3. Shade the half plane above the line if the inequality is "greater than." Shade the half plane under the line if the inequality is "less than."

Example A

Graph the inequality $y \geq 2x-3$.

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x-3$; then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.



Example B

Graph the inequality $5x-2y > 4$.

Solution

First we need to rewrite the inequality in slope-intercept form:

$$-2y > -5x + 4$$
$$y < \frac{5}{2}x - 2$$

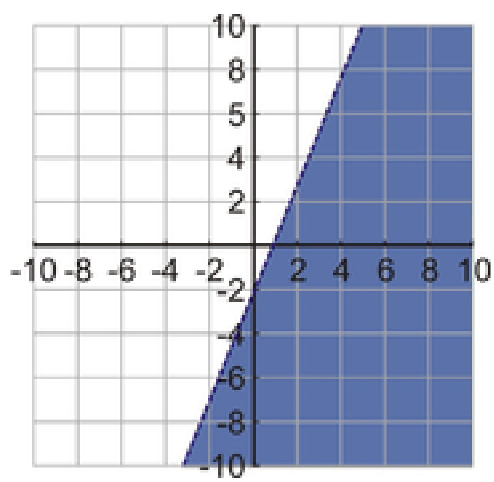
Notice that the inequality sign changed direction because we divided by a negative number.

To graph the equation, we can make a table of values:

TABLE 4.18:

x	y
-2	$\frac{5}{2}(-2) - 2 = -7$
0	$\frac{5}{2}(0) - 2 = -2$
2	$\frac{5}{2}(2) - 2 = 3$

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example C

A retailer sells two types of coffee beans. One type costs $9 per pound and the other type costs $7 per pound. Find all the possible amounts of the two different coffee beans that can be mixed together to get a quantity of coffee beans costing $8.50 or less.

Solution

Let $x =$ weight of $9 per pound coffee beans in pounds.

Let $y =$ weight of $7 per pound coffee beans in pounds.

The cost of a pound of coffee blend is given by $9x + 7y$.

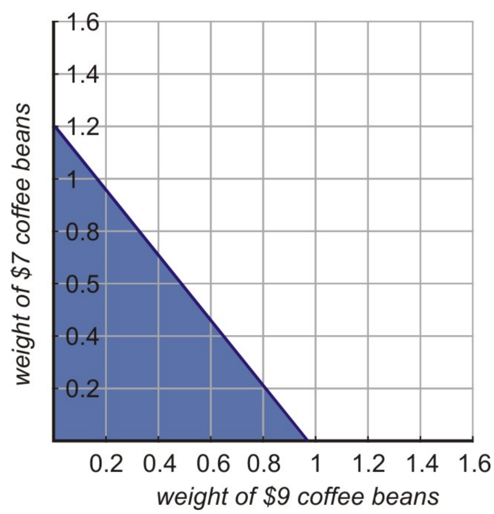
We are looking for the mixtures that cost $8.50 or less. We write the inequality $9x + 7y \leq 8.50$.

Since this inequality is in standard form, it's easiest to graph it by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $7y = 8.50$ or $y = \frac{8.50}{7} \approx 1.21$. When $y = 0$, we have $9x = 8.50$ or $x = \frac{8.50}{9} \approx 0.94$. We can then graph the line that includes those two points.

Now we have to figure out which side of the line to shade. In $y-$intercept form, we shade the area **below** the line when the inequality is "less than." But in standard form that's not always true. We could convert the inequality to $y-$intercept form to find out which side to shade, but there is another way that can be easier.

The other method, which works for any linear inequality in any form, is to plug a random point into the inequality and see if it makes the inequality true. Any point that's not on the line will do; the point (0, 0) is usually the most convenient.

In this case, plugging in 0 for x and y would give us $9(0) + 7(0) \leq 8.50$, which is true. That means we should shade the half of the plane that includes (0, 0). If plugging in (0, 0) gave us a false inequality, that would mean that the solution set is the part of the plane that does *not* contain (0, 0).



Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always nonnegative, so points outside the first quadrant don't represent real-world solutions to this problem.

Vocabulary

- For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

- The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

$>$The solution set is the half plane above the line.

$\geq$ The solution set is the half plane above the line and also all the points on the line.

$<$The solution set is the half plane below the line.

$\leq$ The solution set is the half plane below the line and also all the points on the line.

Guided Practice

Julius has a job as an appliance salesman. He earns a commission of $60 for each washing machine he sells and $130 for each refrigerator he sells. How many washing machines and refrigerators must Julius sell in order to make $1000 or more in commissions?

Solution

Let $x =$ number of washing machines Julius sells.

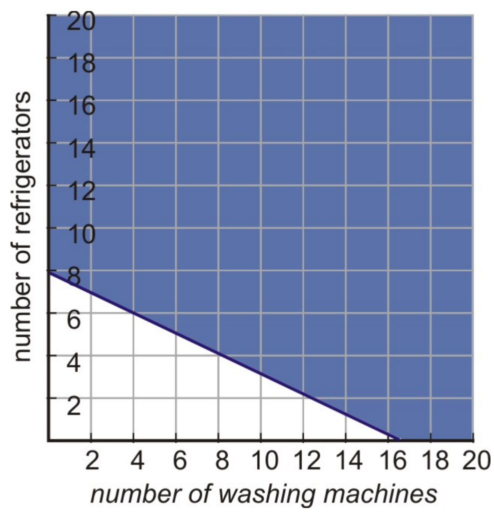
Let $y =$ number of refrigerators Julius sells.

The total commission is $60x + 130y$.

We're looking for a total commission of $1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Once again, we can do this most easily by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $130y = 1000$, or $y = \frac{1000}{30} \approx 7.69$. When $y = 0$, we have $60x = 1000$, or $x = \frac{1000}{60} \approx 16.67$.

We draw a solid line connecting those points, and shade above the line because the inequality is "greater than." We can check this by plugging in the point (0, 0): selling 0 washing machines and 0 refrigerators would give Julius a commission of $0, which is *not* greater than or equal to $1000, so the point (0, 0) is *not* part of the solution; instead, we want to shade the side of the line that does *not* include it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius's commission should be non-negative.

Practice

Graph the following inequalities on the coordinate plane.

1. $y \leq 4x + 3$
2. $y > -\frac{x}{2} - 6$
3. $3x - 4y \geq 12$
4. $x + 7y < 5$
5. $6x + 5y > 1$
6. $y + 5 \leq -4x + 10$
7. $x - \frac{1}{2}y \geq 5$
8. $6x + y < 20$
9. $30x + 5y < 100$

10. Remember what you learned in the last chapter about families of lines.
    a. What do the graphs of $y > x+2$ and $y < x+5$ have in common?
    b. What do you think the graph of $x+2 < y < x+5$ would look like?
11. How would the answer to problem 6 change if you subtracted 2 from the right-hand side of the inequality?
12. How would the answer to problem 7 change if you added 12 to the right-hand side?
13. How would the answer to problem 8 change if you flipped the inequality sign?
14. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and nighttime minutes could you use in one week if you wanted to pay less than $20?
15. Suppose you are graphing the inequality $y > 5x$.
    a. Why can't you plug in the point (0, 0) to tell you which side of the line to shade?
    b. What happens if you do plug it in?
    c. Try plugging in the point (0, 1) instead. Now which side of the line should you shade?
16. A theater wants to take in at least $2000 for a certain matinee. Children's tickets cost $5 each and adult tickets cost $10 each.
    a. If x represents the number of adult tickets sold and y represents the number of children's tickets, write an inequality describing the number of tickets that will allow the theater to meet their minimum take.
    b. If 100 children's tickets and 100 adult tickets have already been sold, what inequality describes how many *more* tickets of both types the theater needs to sell?
    c. If the theater has only 300 seats (so only 100 are still available), what inequality describes the *maximum* number of additional tickets of both types the theater can sell?