CHAPTER 5 Chapter 5: Systems

Chapter Outline

5.1 Solving Systems with One Solution Using Graphing

Here you'll learn how to graph lines to identify the unique solution to a system of linear equations.

Guidance

In this lesson we will be using various techniques to graph the pairs of lines in systems of linear equations to identify the point of intersection or the solution to the system. It is important to use graph paper and a straightedge to graph the lines accurately. Also, you are encouraged to check your answer algebraically as described in the previous lesson.

Example A

Graph and solve the system:

$$y = -x + 1$$
$$y = \frac{1}{2}x - 2$$

Solution:

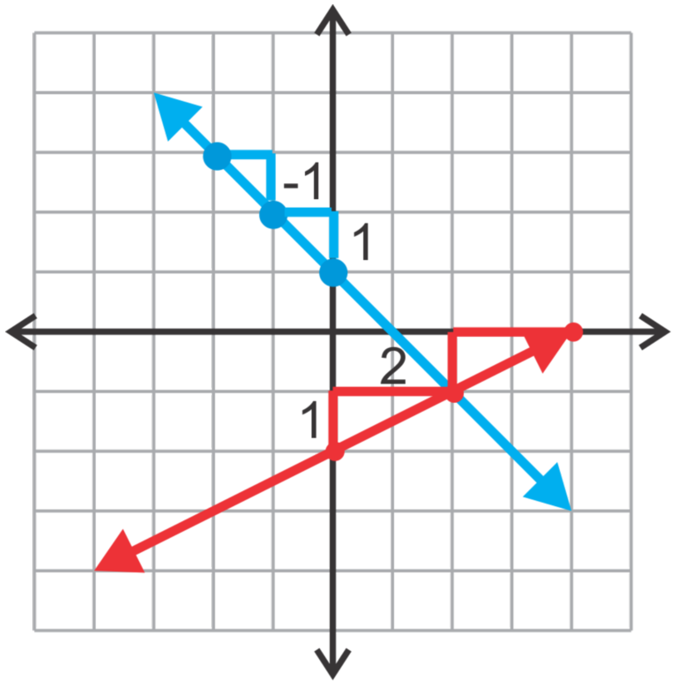
Since both of these equations are written in slope intercept form, we can graph them easily by plotting the $y-$intercept point and using the slope to locate additional points on each line.

The equation, $y = -x + 1$, graphed in **blue**, has $y-$intercept 1 and slope $-\frac{1}{1}$.

The equation, $y = \frac{1}{2}x - 2$, graphed in **red**, has $y-$intercept -2 and slope $\frac{1}{2}$.

Now that both lines have been graphed, the intersection is observed to be the point (2, -1).

Check this solution algebraically by substituting the point into both equations.



Equation 1: $y = -x + 1$, making the substitution gives: $(-1) = (-2) + 1$.

Equation 2: $y = \frac{1}{2}x - 2$, making the substitution gives: $-1 = \frac{1}{2}(2) - 2$.

(2, -1) is the solution to the system.

Example B

Graph and solve the system:

$$\begin{aligned} 3x + 2y &= 6 \\ y &= -\frac{1}{2}x - 1 \end{aligned}$$

Solution: This example is very similar to the first example. The only difference is that equation 1 is not in slope intercept form. We can either solve for y to put it in slope intercept form or we can use the intercepts to graph the equation. To review using intercepts to graph lines, we will use the latter method.

Recall that the $x-$intercept can be found by replacing y with zero and solving for x:

$$\begin{aligned} 3x + 2(0) &= 6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Similarly, the $y-$intercept is found by replacing x with zero and solving for y:

$$\begin{aligned} 3(0) + 2y &= 6 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

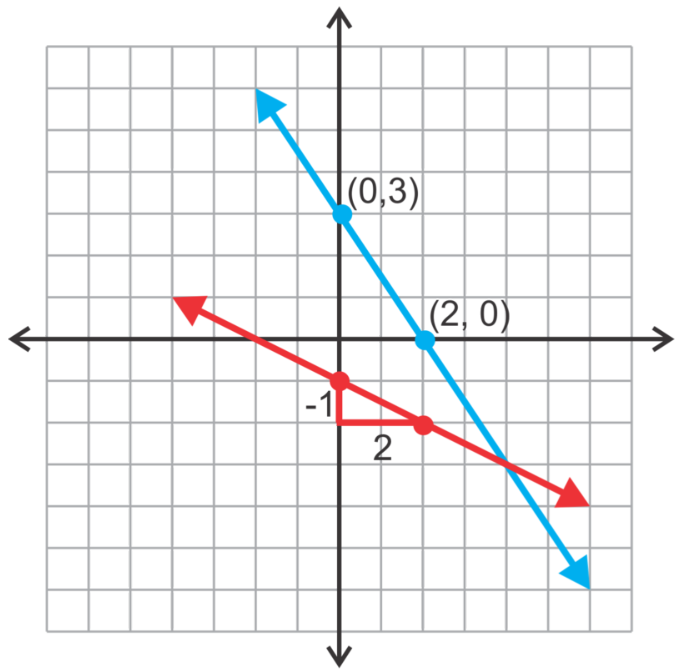
We have two points, (2, 0) and (0, 3) to plot and graph this line. Equation 2 can be graphed using the $y-$intercept and slope as shown in Example A.

Now that both lines are graphed we observe that their intersection is the point (4, -3).

Finally, check this solution by substituting it into each of the two equations.

Equation 1: $3x+2y=6; 3(4)+2(-3)=12-6=6$

Equation 2: $y=-\frac{1}{2}x-1; -3=-\frac{1}{2}(4)-1$



Example C

In this example we will use technology to solve the system:

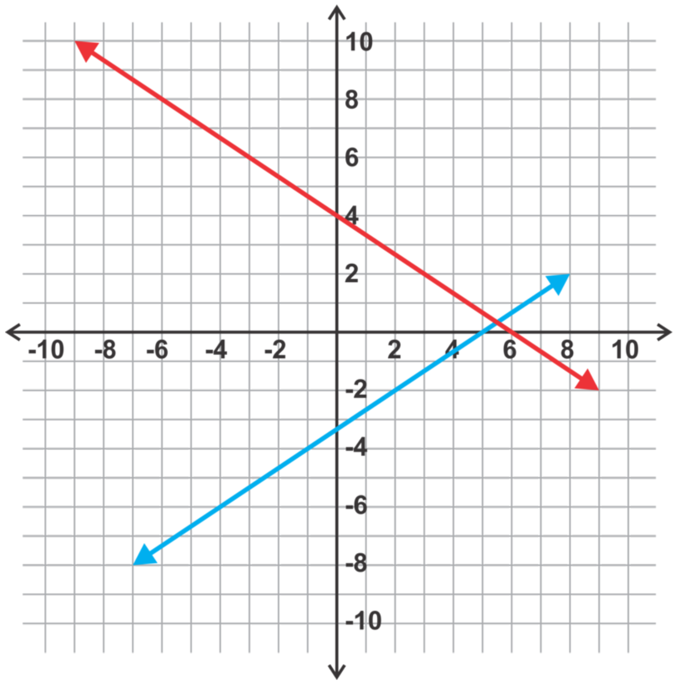
$$\begin{aligned} 2x-3y&=10 \\ y&=-\frac{2}{3}x+4 \end{aligned}$$

This process may vary somewhat based on the technology you use. All directions here can be applied to the TI-83 or 84 (plus, silver, etc) calculators.

Solution: The first step is to graph these equations on the calculator. The first equation must be rearranged into slope intercept form to put in the calculator.

$$\begin{aligned} 2x-3y&=10 \\ -3y&=-2x+10 \\ y&=\frac{-2x+10}{-3} \\ y&=\frac{2}{3}x-\frac{10}{3} \end{aligned}$$

The graph obtained using the calculator should look like this:



The first equation, $y = \frac{2}{3}x - \frac{10}{3}$, is graphed in **blue**. The second equation, $y = -\frac{2}{3}x + 4$, is graphed in **red**.

The solution does not lie on the "grid" and is therefore difficult to observe visually. With technology we can calculate the intersection. If you have a TI-83 or 84, use the CALC menu, select INTERSECT. Then select each line by pressing ENTER on each one. The calculator will give you a "guess." Press ENTER one more time and the calculator will then calculate the intersection of (5.5, .333...). We can also write this point as $\left(\frac{11}{2}, \frac{1}{3}\right)$. Check the solution algebraically.

Equation 1: $2x - 3y = 10; 2\left(\frac{11}{2}\right) - 3\left(\frac{1}{3}\right) = 11 - 1 = 10$

Equation 2: $y = -\frac{2}{3}x + 4; -\frac{2}{3}\left(\frac{11}{2}\right) + 4 = -\frac{11}{3} + \frac{12}{3} = \frac{1}{3}$

If you do not have a TI-83 or 84, the commands might be different. Check with your teacher.

Intro Problem Revisit The system of linear equations represented by this situation is:

$$2c1 + 4c2 = 70$$
$$c1 + 5c2 = 50$$

If you plot both of these linear equations on the same graph, you find that the point of intersection is (25, 5). Therefore coin one has a value of 25 cents and coin two has a value of 5 cents.

Guided Practice

Solve the following systems by graphing. Use technology for problem 3.

1.

$$y = 3x - 4$$
$$y = 2$$

2.

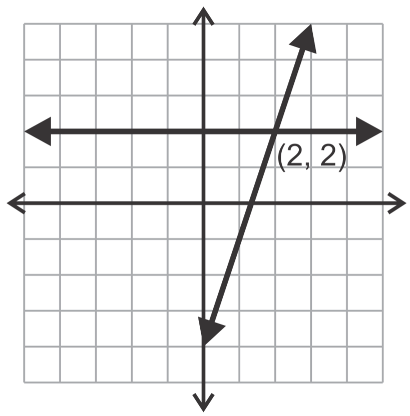
$$2x - y = -4$$
$$2x + 3y = -12$$

3.

$$5x + y = 10$$
$$y = \frac{2}{3}x - 7$$

Answers

1.



The first line is in slope intercept form and can be graphed accordingly.

The second line is a horizontal line through (0, 2).

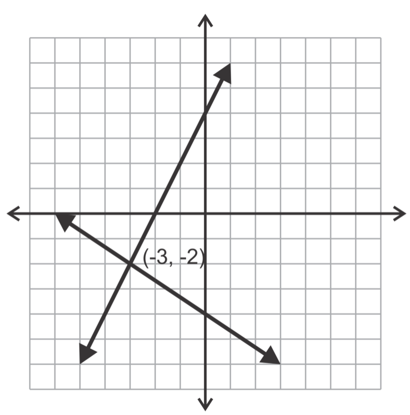
The graph of the two equations is shown below. From this graph the solution appears to be (2, 2).

Checking this solution in each equation verifies that it is indeed correct.

Equation 1: $2 = 3(2) - 4$

Equation 2: $2 = 2$

2.



Neither of these equations is in slope intercept form. The easiest way to graph them is to find their intercepts as shown in Example B.

Equation 1: Let $y = 0$ to find the $x-$intercept.

$$\begin{aligned} 2x - y &= -4 \\ 2x - 0 &= -4 \\ x &= -2 \end{aligned}$$

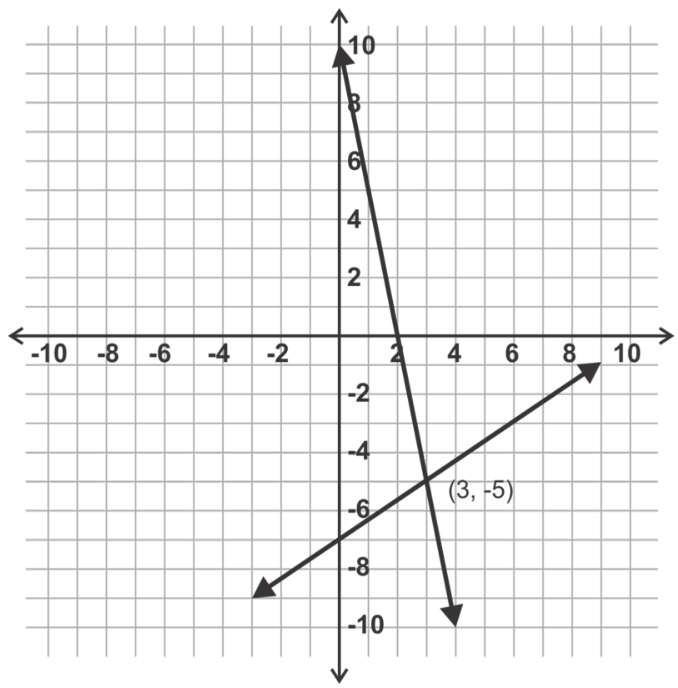
Now let $x = 0$, to find the $y-$intercept.

$$\begin{aligned} 2x - y &= -4 \\ 2(0) - y &= -4 \\ y &= 4 \end{aligned}$$

Now we can use (-2, 0) and (0, 4) to graph the line as shown in the diagram. Using the same process, the intercepts for the second line can be found to be (-6, 0) and (0, -4).

Now the solution to the system can be observed to be (-3, -2). This solution can be verified algebraically as shown in the first problem.

3.



The first equation here must be rearranged to be $y = -5x + 10$ before it can be entered into the calculator. The second equation can be entered as is.

Using the calculate menu on the calculator the solution is (3, -5).

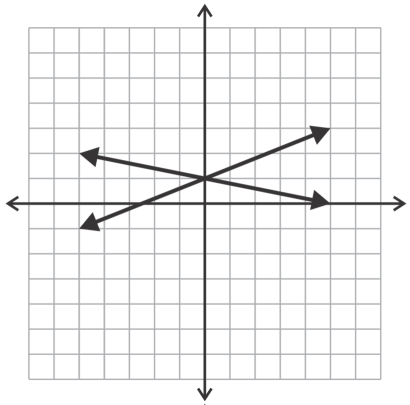
Remember to verify this solution algebraically as a way to check your work.

Practice

Match the system of linear equations to its graph and state the solution.

1.

$$3x+2y=-2$$
$$x-y=-4$$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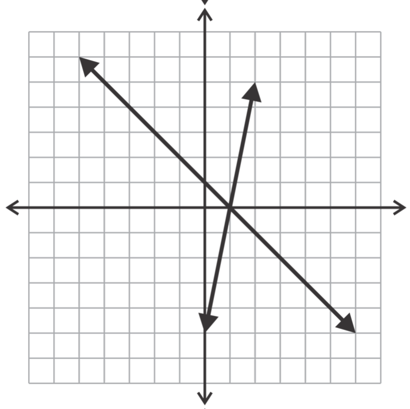
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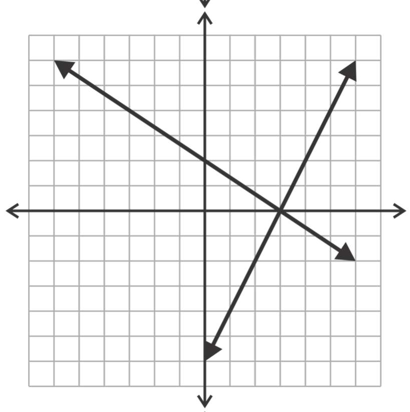
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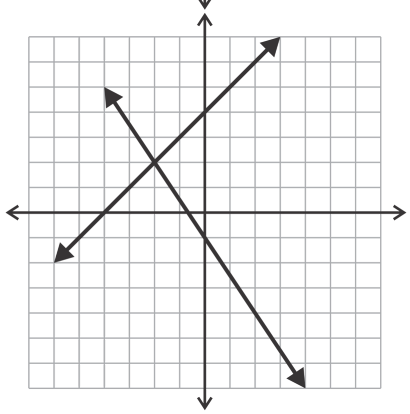
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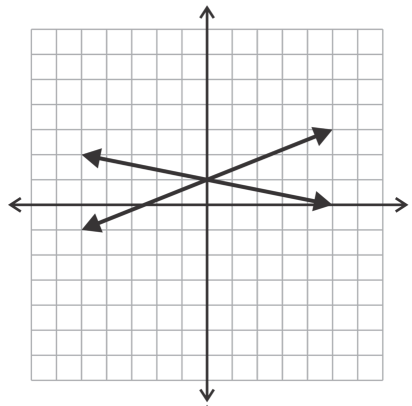
2.

$$2x - y = 6$$
$$2x + 3y = 6$$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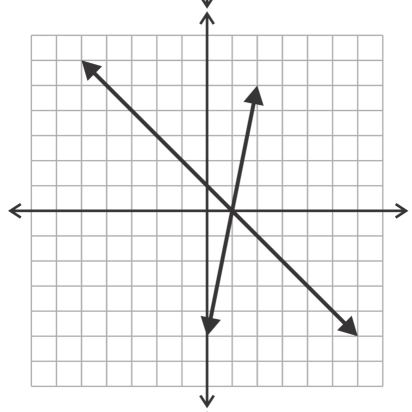
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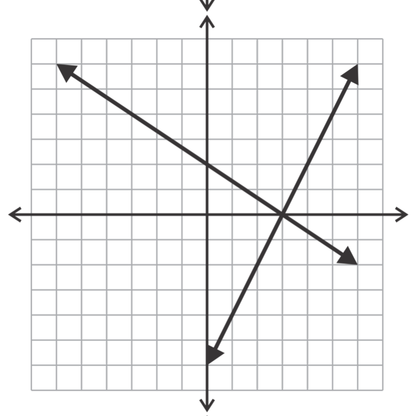
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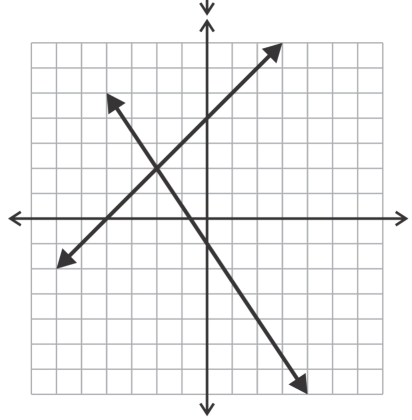
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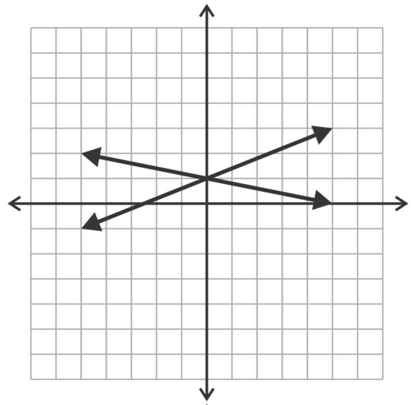
3.

$$2x - 5y = -5$$
$$x + 5y = 5$$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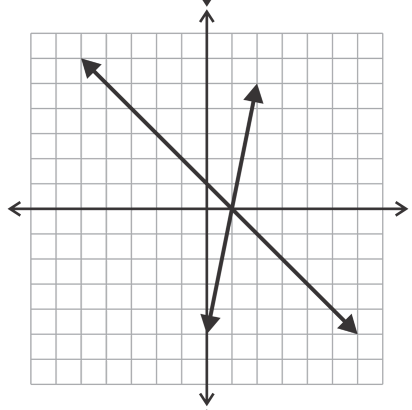
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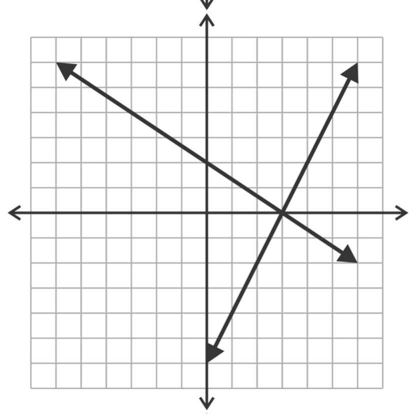
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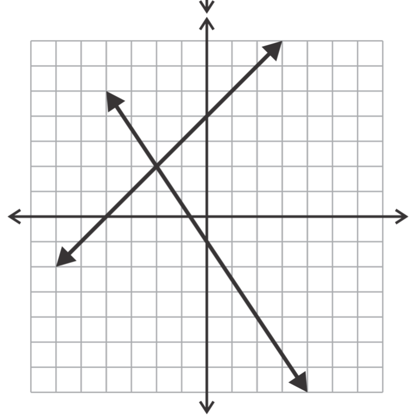
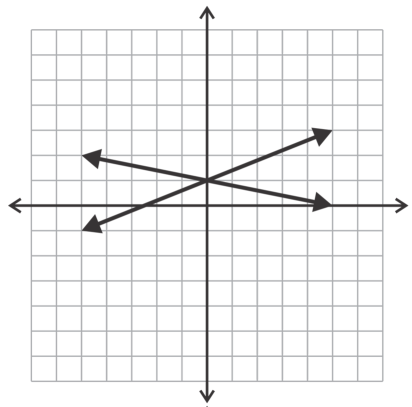
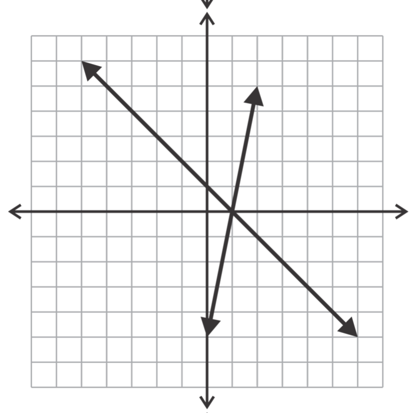
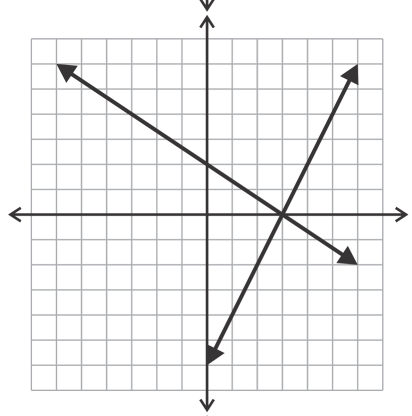

d.

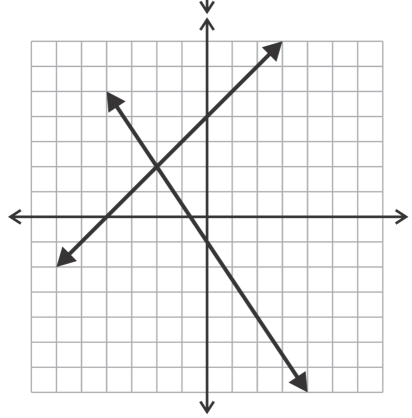
4.

$$y = 5x - 5$$
$$y = -x + 1$$

a. 

b. 

c. 

d. 

Solve the following linear systems by graphing. Use graph paper and a straightedge to insure accuracy. You are encouraged to verify your answer algebraically.

5.

$$
\begin{aligned}
y &= -\frac{2}{5}x+1 \\
y &= \frac{3}{5}x-4
\end{aligned}
$$

6.

$$
\begin{aligned}
y &= -\frac{2}{3}x+4 \\
y &= 3x-7
\end{aligned}
$$

7.

$$
\begin{aligned}
y &= -2x+1 \\
x-y &= -4
\end{aligned}
$$

8.

$$
\begin{aligned}
3x+4y &= 12 \\
x-4y &= 4
\end{aligned}
$$

9.

$$
\begin{aligned}
7x-2y &= -4 \\
y &= -5
\end{aligned}
$$

10.

$$
\begin{aligned}
x-2y &= -8 \\
x &= -3
\end{aligned}
$$

Solve the following linear systems by graphing using technology. Solutions should be rounded to the nearest hundredth as necessary.

11.

$$y = \frac{3}{7}x + 11$$
$$y = -\frac{13}{2}x - 5$$

12.

$$y = 0.95x - 8.3$$
$$2x + 9y = 23$$

13.

$$15x - y = 22$$
$$3x + 8y = 15$$

Use the following information to complete exercises 14-17.

Clara and her brother, Carl, are at the beach for vacation. They want to rent bikes to ride up and down the boardwalk. One rental shop, Bargain Bikes, advertises rates of \$5 plus \$1.50 per hour. A second shop, Frugal Wheels, advertises a rate of \$6 plus \$1.25 an hour.

14. How much does it cost to rent a bike for one hour from each shop? How about 10 hours?
15. Write equations to represent the cost of renting a bike from each shop. Let x represent the number of hours and y represent the total cost.
16. Solve your system to figure out when the cost is the same.
17. Clara and Carl want to rent the bikes for about 3 hours. Which shop should they use?

5.2 Substitution Method for Systems of Equations

Here you'll learn how to solve systems of linear equations algebraically using the substitution method.

When you solve a system of consistent and independent equations by graphing, a single ordered pair is the solution. The ordered pair satisfies both equations and the point is the intersection of the graphs of the linear equations. The coordinates of this point of intersection are not always integers. How can you algebraically solve a system of equations like the one below?

$$\begin{Bmatrix} 2x+3y=13 \\ y=4x-5 \end{Bmatrix}$$

Guidance

A 2×2 system of linear equations can be solved algebraically by the substitution method. In order to use this method, follow these steps:

1. Solve one of the equations for one of the variables.
2. Substitute that expression into the remaining equation. The result will be a linear equation, with one variable, that can be solved.
3. Solve the remaining equation.
4. Substitute the solution into the other equation to determine the value of the other variable.
5. The solution to the system is the intersection point of the two equations and it represents the coordinates of the ordered pair.

Example A

Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} 3x+y=1 \\ 2x+5y=18 \end{Bmatrix}$$

Solution: To begin, solve one of the equations in terms of one of the variables. This step is simplified if one of the equations has one variable with a coefficient that is either +1 or –1. In the above system the first equation has 'y' with a coefficient of 1.

$$\begin{aligned} &3x+y=1 \\ &3x-3x+y=1-3x \\ &\boxed{y=1-3x} \end{aligned}$$

Substitute $(1-3x)$ into the second equation for 'y'.

$$\begin{aligned}2x+5y&=18\\2x+5(1-3x)&=18\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}&2x+5-15x=18\\&-13x+5=18\\&-13x+5-5=18-5\\&-13x=13\\&\frac{-13x}{-13}=\frac{13}{-13}\\&\frac{\cancel{-13}x}{\cancel{-13}}=\frac{\overset{-1}{\cancel{-13}}}{\cancel{-13}}\\&\boxed{x=-1}\end{aligned}$$

Substitute –1 for x into the equation

$$\boxed{y=1-3x}$$

.

$$\begin{aligned}&y=1-3x\\&y=1-3(-1)\end{aligned}$$

$$\begin{aligned}&y=1+3\\&\boxed{y=4}\end{aligned}$$

The solution is (–1, 4). This represents the intersection point of the lines if the equations were graphed on a Cartesian grid. Another way to write 'the lines intersect at (–1, 4)' is:

$$\boxed{\text{Line1}}:3x+y=1$$

$$\boxed{\text{Line2}}:2x+5y=18$$

Line 1 intersects Line 2 at (–1, 4)

$$\begin{array}{c}\text{at}\\\uparrow\\\boxed{l_1\cap l_2@(-1,4)}\\\downarrow\\\text{intersects}\end{array}$$

Example B

Solve the following system of linear equations by substitution:

$$\left\{\begin{array}{c} 8x-3y=6 \\ 6x+12y=-24 \end{array}\right\}$$

Solution: There is no variable that has a coefficient of +1 or of –1. However, the second equation has coefficients and a constant that are multiples of 6. The second equation will be solved for the variable 'x'.

$$\begin{aligned}
&6x+12y=-24\\
&6x+12y-12y=-24-12y\\
&6x=-24-12y\\
&\frac{6x}{6}=\frac{-24}{6}-\frac{12y}{6}\\
&\frac{\cancel{6}x}{\cancel{6}}=\frac{\overset{-4}{\cancel{-24}}}{\cancel{6}}-\frac{\overset{2}{\cancel{12}}y}{\cancel{6}}\\
&\boxed{x=-4-2y}
\end{aligned}$$

Substitute $(-4-2y)$ into the first equation for 'x'.

$$\begin{aligned}
8x-3y&=6\\
8(-4-2y)-3y&=6
\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}
&-32-16y-3y=6\\
&-32-19y=6\\
&-32+32-19y=6+32\\
&-19y=38\\
&\frac{-19y}{-19}=\frac{38}{-19}\\
&\frac{\cancel{-19}y}{\cancel{-19}}=\frac{\overset{-2}{\cancel{38}}}{\cancel{-19}}\\
&\boxed{y=-2}
\end{aligned}$$

Substitute –2 for y into the equation

$$\boxed{x=-4-2y}$$

$$\begin{aligned}
x&=-4-2y\\
x&=-4-2(-2)
\end{aligned}$$

$$x = -4 + 4$$
$$\boxed{x = 0}$$
$$\boxed{l_1 \cap l_2 @ (0, -2)}$$

Example C

Jason, who is a real computer whiz, decided to set up his own server and to sell space on his computer so students could have their own web pages on the Internet. He devised two plans. One plan charges a base fee of $25.00 plus $0.50 every month. His other plan has a base fee of $5.00 plus $1 per month.

i) Write an equation to represent each plan.

ii) Solve the system of equations.

Solution: Both plans deal with the cost of buying space from Jason's server. The cost involves a base fee and a monthly rate. The equations for the plans are:

- $y = 0.50x + 25$
- $y = 1x + 5$

where 'y' represents the **cost** and 'x' represents the **number of months.** Both equations are equal to 'y'. Therefore, the expression for y can be substituted for the y in the other equation.

$$\left\{\begin{array}{c} y = 0.50x + 25 \\ y = 1x + 5 \end{array}\right\}$$

$$0.50x + 25 = 1x + 5$$
$$0.50x + 25 - 25 = 1x + 5 - 25$$
$$0.50x = 1x - 20$$
$$0.50x - 1x = 1x - 1x - 20$$
$$-0.50x = -20$$
$$\frac{-0.50x}{-0.50} = \frac{-20}{-0.50}$$
$$\frac{\cancel{-0.50}x}{\cancel{-0.50}} = \frac{\overset{40}{\cancel{-20}}}{\cancel{-0.50}}$$
$$\boxed{x = 40 \text{ } months}$$

Since the equations were equal, the value for 'x' can be substituted into either of the original equations. The result will be the same.

$$y = 1x + 5$$
$$y = 1(40) + 5$$

$$y = 40 + 5$$
$$\boxed{y = 45\ dollars}$$
$$\boxed{l_1 \cap l_2 @ (40, 45)}.$$

Concept Problem Revisited

When graphing is not a feasible method for solving a system, you can solve by substitution:

$$\begin{Bmatrix} 2x + 3y = 13 \\ y = 4x - 5 \end{Bmatrix}$$

The second equation is solved in terms of the variable 'y'. The expression $(4x+5)$ can be used to replace 'y' in the first equation.

$$2x + 3y = 13$$
$$2x + 3(4x - 5) = 13$$

The equation now has one variable. Apply the distributive property.

$2x + 12x - 15 = 13$

Combine like terms to simplify the equation.

$14x - 15 = 13$

Solve the equation.

$$14x - 15 + 15 = 13 + 15$$
$$14x = 28$$
$$\frac{14x}{14} = \frac{28}{14}$$
$$\frac{\cancel{14}x}{\cancel{14}} = \frac{\overset{2}{\cancel{28}}}{\cancel{14}}$$
$$\boxed{x = 2}$$

To determine the value of 'y', substitute this value into the equation $y = 4x - 5$.

$$y = 4x - 5$$
$$y = 4(2) - 5$$

$$y = 8 - 5$$
$$\boxed{y = 3}$$

The solution is (2, 3). This represents the intersection point of the lines if the equations were graphed on a Cartesian grid.

Vocabulary

Substitution Method

The ***substitution method*** is a way of solving a system of linear equations algebraically. The substitution method involves solving an equation for a variable and substituting that expression into the other equation.

Guided Practice

1. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} x = 2y + 1 \\ x = 4y - 3 \end{Bmatrix}$$

2. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} 2x + y = 3 \\ 3x + 2y = 12 \end{Bmatrix}$$

.

3. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} \frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\ -\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4} \end{Bmatrix}$$

Answers:

1.

$$\begin{Bmatrix} x = 2y + 1 \\ x = 4y - 3 \end{Bmatrix}$$

Both equations are equal to the variable 'x'. Set $(2y+1) = (4y-3)$
$2y + 1 = 4y - 3$
Solve the equation.

$$\begin{aligned}
&2y + 1 - 1 = 4y - 3 - 1 \\
&2y = 4y - 4 \\
&2y - 4y = 4y - 4y - 4 \\
&-2y = -4 \\
&\frac{-2y}{-2} = \frac{-4}{-2} \\
&\frac{\cancel{-2}y}{\cancel{-2}} = \frac{\cancel{-4}^{\,2}}{\cancel{-2}} \\
&\boxed{y = 2}
\end{aligned}$$

Substitute this value for 'y' into one of the original equations.

$$\begin{aligned}
&x = 2y + 1 \\
&x = 2(2) + 2
\end{aligned}$$

$$x = 4 + 1$$
$$\boxed{x = 5}$$
$$\boxed{l_1 \cap l_2 @ (5,2)}$$

2.

$$\begin{Bmatrix} 2x + y = 3 \\ 3x + 2y = 12 \end{Bmatrix}$$

The first equation has the variable 'y' with a coefficient of 1. Solve the equation in terms of 'y'.

$$2x + y = 3$$
$$2x - 2x + y = 3 - 2x$$
$$\boxed{y = 3 - 2x}$$

Substitute $(3-2x)$ into the second equation for 'y'.

$$3x + 2y = 12$$
$$3x + 2(3 - 2x) = 12$$

Apply the distributive property and solve the equation.

$$3x + 6 - 4x = 12$$
$$6 - x = 12$$
$$6 - 6 - x = 12 - 6$$
$$-x = 6$$
$$\frac{-x}{-1} = \frac{6}{-1}$$
$$\frac{\cancel{-1}x}{\cancel{-1}} = \frac{\cancelto{-6}{6}}{\cancel{-1}}$$
$$\boxed{x = -6}$$

Substitute this value for 'x' into the equation $y = 3 - 2x$.

$$y = 3 - 2x$$
$$y = 3 - 2(-6)$$

$$y = 3 + 12$$
$$\boxed{y = 15}$$
$$\boxed{l_1 \cap l_2 @ (-6, 15)}$$

3.

$$\begin{Bmatrix} \frac{2}{5}m+\frac{3}{4}n=\frac{5}{2} \\ -\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4} \end{Bmatrix}$$

Begin by multiplying each equation by the LCM of the denominators to simplify the system.
$\frac{2}{5}m+\frac{3}{4}n=\frac{5}{2}$ The LCM for the denominators is 20.

$$\begin{aligned} 20\left(\frac{2}{5}\right)m+20\left(\frac{3}{4}\right)n &= 20\left(\frac{5}{2}\right) \\ \overset{4}{\cancel{20}}\left(\frac{2}{\cancel{5}}\right)m+\overset{5}{\cancel{20}}\left(\frac{3}{\cancel{4}}\right)n &= \overset{10}{\cancel{20}}\left(\frac{5}{\cancel{2}}\right) \\ 8m+15n &= 50 \\ 8m+15n &= 50 \end{aligned}$$

$-\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4}$ The LCM for the denominators is 12.

$$\begin{aligned} -12\left(\frac{2}{3}\right)m+12\left(\frac{1}{2}\right)n &= 12\left(\frac{3}{4}\right) \\ -\overset{4}{\cancel{12}}\left(\frac{2}{\cancel{3}}\right)m+\overset{6}{\cancel{12}}\left(\frac{1}{\cancel{2}}\right)n &= \overset{3}{\cancel{12}}\left(\frac{3}{\cancel{4}}\right) \\ -8m+6n &= 9 \\ -8m+6n &= 9 \end{aligned}$$

The two equations that need to be solved by substitution are:

$$\begin{Bmatrix} 8m+15n=50 \\ -8m+6n=9 \end{Bmatrix}$$

Neither of the equations have a variable with a coefficient of 1 nor does one equation have coefficients that are multiples of a given coefficient. Solve the first equation in terms of 'm'.

$$\begin{aligned} &8m+15n=50 \\ &8m+15n-15n=50-15n \\ &8m=50-15n \\ &\frac{8m}{8}=\frac{50}{8}-\frac{15n}{8} \\ &\frac{\cancel{8}m}{\cancel{8}}=\frac{50}{8}-\frac{15n}{8} \\ &m=\frac{25}{4}-\frac{15}{8}n \\ &\boxed{m=\frac{25}{4}-\frac{15}{8}n} \end{aligned}$$

Substitute this value for 'm' into the second equation.

$$\begin{aligned} -8m+6n &= 9 \\ -8\left(\frac{25}{4}-\frac{15}{8}n\right)+6n &= 9 \end{aligned}$$

Apply the distributive property and solve the equation.

$$-\frac{200}{4}+\frac{120}{8}n+6n=9$$
$$-\frac{\overset{50}{\cancel{200}}}{\cancel{4}}+\frac{\overset{15}{\cancel{120}}}{\cancel{8}}n+6n=9$$
$$-50+15n+6n=9$$
$$-50+21n=9$$
$$-50+50+21n=9+50$$
$$21n=59$$
$$\frac{21n}{21}=\frac{59}{21}$$
$$\frac{\cancel{21}n}{\cancel{21}}=\frac{59}{21}$$
$$\boxed{n=\frac{59}{21}}$$

Substitute this value into the equation that has been solved in terms of 'm' or into one of the original equations or into one of the new equations that resulted from multiplying by the LCM.
Whichever substitution is performed, the same result will occur.

$$m=\frac{25}{4}-\frac{15}{8}n$$
$$m=\frac{25}{4}-\frac{15}{8}\left(\frac{59}{21}\right)$$
$$m=\frac{25}{4}-\frac{885}{168}$$

A common denominator is required to subtract the fractions.

$$\begin{array}{r} 42 \\ 4\overline{)168} \\ \underline{-16\downarrow} \\ 8 \\ \underline{-\ \ 8} \\ 0 \end{array}$$

Multiply $\frac{25}{4}$ *by* $\frac{42}{42}$:

$$m = \frac{42}{42}\left(\frac{25}{4}\right) - \frac{885}{168}$$
$$m = \frac{1050}{168} - \frac{885}{168}$$
$$m = \frac{165}{168}$$
$$m = \frac{55}{56}$$
$$\boxed{m = \frac{55}{56}}$$
$$\boxed{l_1 \cap l_2 @ \left(\frac{55}{56}, \frac{59}{21}\right)}$$

Practice

Solve the following systems of linear equations using the substitution method.

1.

$$\left\{\begin{array}{c} y = 3x \\ 5x - 2y = 1 \end{array}\right\}$$

2.

$$\left\{\begin{array}{c} y = 3x + 1 \\ 2x - y = 2 \end{array}\right\}$$

3.

$$\left\{\begin{array}{c} x = 2y \\ x = 3y - 3 \end{array}\right\}$$

4.

$$\left\{\begin{array}{c} x - y = 6 \\ 6x - y = 40 \end{array}\right\}$$

5.

$$\left\{\begin{array}{c} x + y = 6 \\ x + 3(y + 2) = 10 \end{array}\right\}$$

6.

$$\begin{Bmatrix} 2x+y=5 \\ 3x-4y=2 \end{Bmatrix}$$

7.

$$\begin{Bmatrix} 5x-2y=-4 \\ 4x+y=-11 \end{Bmatrix}$$

8.

$$\begin{Bmatrix} 3y-x=-10 \\ 3x+4y=-22 \end{Bmatrix}$$

9.

$$\begin{Bmatrix} 4e+2f=-2 \\ 2e-3f=1 \end{Bmatrix}$$

10.

$$\begin{Bmatrix} \frac{1}{4}x+y=-\frac{7}{2} \\ \frac{1}{2}x-\frac{1}{4}y=1 \end{Bmatrix}$$

11.

$$\begin{Bmatrix} x=-4+y \\ x=3y-6 \end{Bmatrix}$$

12.

$$\begin{Bmatrix} 3y-2x=-3 \\ 3x-3y=6 \end{Bmatrix}$$

13.

$$\begin{Bmatrix} 2x=5y-12 \\ 3x+5y=7 \end{Bmatrix}$$

14.

$$\begin{Bmatrix} 3y = 2x - 5 \\ 2x = y + 3 \end{Bmatrix}$$

15.

$$\begin{Bmatrix} \frac{x+y}{3} + \frac{x-y}{2} = \frac{25}{6} \\ \frac{x+y-9}{2} = \frac{y-x-6}{3} \end{Bmatrix}$$

5.3 Elimination Method for Systems of Equations

Here you'll learn how to solve a system of linear equations algebraically using the elimination method.

When you solve a system of consistent and independent equations by graphing, a single ordered pair is the solution. The ordered pair satisfies both equations and the point is the intersection of the graphs of the linear equations. The coordinates of this point of intersection are not always integers. Therefore, some method has to be used to determine the values of the coordinates. How can you algebraically solve the system of equations below without first rewriting the equations?

$$\begin{Bmatrix} 2x+3y=5 \\ 3x-3y=10 \end{Bmatrix}$$

Guidance

A 2×2 system of linear equations can be solved algebraically by the elimination method. To use this method you must write an equivalent system of equations such that, when two of the equations are added or subtracted, one of the variables is eliminated. The solution is the intersection point of the two equations and it represents the coordinates of the ordered pair. This method is demonstrated in the examples.

Example A

Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} 3y=2x-5 \\ 2x=y+3 \end{Bmatrix}$$

Solution: To begin, set up the equations so that they are in the format

$$\begin{Bmatrix} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{Bmatrix}$$

$$\begin{aligned} 3y &= 2x-5 \\ -2x+3y &= 2x-2x-5 \\ -2x+3y &= -5 \end{aligned} \qquad \begin{aligned} 2x &= y+3 \\ 2x-y &= y-y+3 \\ 2x-y &= 3 \end{aligned}$$

Solve the formatted system of equations:

$$\begin{Bmatrix} -2x+3y=-5 \\ 2x-y=3 \end{Bmatrix}$$

Both equations have a term that is $2x$. In the first equation the coefficient of x is a negative two and in the second equation the coefficient of x is a positive two. If the two equations are added, the x variable is eliminated.

$$\begin{aligned}
-\cancel{2x} + 3y &= -5 \\
\cancel{2x} - y &= +3 \\
\hline
2y &= -2 \quad \text{Eliminate the variable } x.
\end{aligned}$$

$$\begin{aligned}
&2y = -2 \quad \text{Solve the equation.} \\
&\frac{2y}{2} = \frac{-2}{2} \\
&\frac{\cancel{2}y}{\cancel{2}} = \frac{\overset{-1}{\cancel{-2}}}{\cancel{2}} \\
&\boxed{y = -1}
\end{aligned}$$

The value of y is –1. This value can now be substituted into one of the original equations to determine the value of x. Remember x is the variable that was eliminated from the system of linear equations.

$$\begin{aligned}
&2x - y = 3 \\
&2x - (-1) = 5 && \text{Substitute in the value for } y. \\
&2x + 1 = 5 && \text{Multiply the value of } x \text{ by the coefficient } (-1). \\
&2x + 1 - 1 = 3 - 1 && \text{Isolate the variable } x. \\
&2x = 2 && \text{Solve the equation.} \\
&\frac{2x}{2} = \frac{2}{2} \\
&\frac{\cancel{2}x}{\cancel{2}} = \frac{\overset{1}{\cancel{2}}}{\cancel{2}} \\
&\boxed{x = 1} \\
&\boxed{l_1 \cap l_2 @ (1, -1)}
\end{aligned}$$

This means "Line 1 intersects Line 2 at the point (1, –1)".

Example B

Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} 2x - 3y = 13 \\ 3x + 4y = -6 \end{Bmatrix}$$

Solution: The coefficients of 'x' are 2 and 3. The coefficients of 'y' are –3 and 4. To eliminate a variable the coefficients must be the same number but with opposite signs. This can be accomplished by multiplying one or both of the equations.

The first step is to choose a variable to eliminate. If the choice is 'x', the least common multiple of 2 and 3 is 6. This means that the equations must be multiplied by 3 and 2 respectively. One of the multipliers must be a negative number so that one of the coefficients of ’x’ will be a negative 6. When this is done, the coefficients of ’x’ will be +6 and –6. The variable will then be eliminated when the equations are added.

Multiply the first equation by negative three.

$$-3(2x-3y=13)$$
$$-6x+9y=-39$$

Multiply the second equation by positive two.

$$2(3x+4y=-6)$$
$$6x+8y=-12$$

Add the two equations.

$$\begin{aligned}
-\cancel{6x}+9y&=-39\\
\cancel{6x}+8y&=-12\\
\hline
17y&=-51 \qquad \text{Solve the equation.}\\
17y&=-51\\
\frac{17y}{17}&=\frac{-51}{17}\\
\frac{\cancel{17}y}{\cancel{17}}&=\frac{\overset{-3}{\cancel{-51}}}{\cancel{17}}\\
\boxed{y=-3}
\end{aligned}$$

Substitute the value for y into one of the original equations.

$$\begin{aligned}
&2x-3y=13\\
&2x-3(-3)=13 && \text{Substitute in the value for } y.\\
&2x+9=13 && \text{Multiply the value of } y \text{ by the coefficient } (-3).\\
&2x+9-9=13-9 && \text{Isolate the variable } x.\\
&2x=4 && \text{Solve the equation.}\\
&\frac{2x}{2}=\frac{4}{2}\\
&\frac{\cancel{2}x}{\cancel{2}}=\frac{\overset{2}{\cancel{4}}}{\cancel{2}}\\
&\boxed{x=2}\\
&\boxed{l_1\cap l_2 @ (2,-3)}
\end{aligned}$$

Example C

Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} \frac{3}{4}x + \frac{5}{4}y = 4 \\ \frac{1}{2}x + \frac{1}{3}y = \frac{5}{3} \end{Bmatrix}$$

Solution: Begin by multiplying each equation by the LCD to create two equations with integers as the coefficients of the variables.

$$\begin{aligned} \frac{3}{4}x + \frac{5}{4}y &= 4 \\ 4\left(\frac{3}{4}\right)x + 4\left(\frac{5}{4}\right)y &= 4(4) \\ \cancel{4}\left(\frac{3}{\cancel{4}}\right)x + \cancel{4}\left(\frac{5}{\cancel{4}}\right)y &= 4(4) \\ 3x + 5y &= 16 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= \frac{5}{3} \\ 6\left(\frac{1}{2}\right)x + 6\left(\frac{1}{3}\right)y &= 6\left(\frac{5}{3}\right) \\ \overset{3}{\cancel{6}}\left(\frac{1}{\cancel{2}}\right)x + \overset{2}{\cancel{6}}\left(\frac{1}{\cancel{3}}\right)y &= \overset{2}{\cancel{6}}\left(\frac{5}{\cancel{3}}\right) \\ 3x + 2y &= 10 \end{aligned}$$

Now solve the following system of equations by elimination:

$$\begin{Bmatrix} 3x + 5y = 16 \\ 3x + 2y = 10 \end{Bmatrix}$$

The coefficients of the 'x' variable are the same –positive three. To change one of them to a negative three, multiply one of the equations by a negative one.

$$\begin{aligned} -1(3x + 5y &= 16) \\ -3x - 5y &= -16 \end{aligned}$$

The two equations can now be added.

$$\begin{aligned} -\cancel{3x} - 5y &= -16 \\ \cancel{3x} + 2y &= \ \ 10 \\ \hline -3y &= \ \ -6 \qquad \text{Solve the equation.} \\ -3y &= -6 \\ \frac{-3y}{-3} &= \frac{-6}{-3} \\ \frac{\cancel{-3}y}{\cancel{-3}} &= \frac{\overset{2}{\cancel{-6}}}{\cancel{-3}} \\ \boxed{y = 2} \end{aligned}$$

Substitute the value for 'y' into one of the original equations.

$$\frac{3}{4}x+\frac{5}{4}y=4$$

$$\frac{3}{4}x+\frac{5}{4}(2)=4$$ Substitute in the value for y.

$$\frac{3}{4}x+\frac{10}{4}=4$$ Multiply the value of y by the coefficient $\left(\frac{5}{4}\right)$.

$$\frac{3}{4}x+\frac{10}{4}-\frac{10}{4}=4-\frac{10}{4}$$ Isolate the variable x.

$$\frac{3}{4}x=\frac{16}{4}-\frac{10}{4}$$

$$\frac{3}{4}x=\frac{6}{4}$$ Multiply both sides by 4.

$$4\left(\frac{3}{4}x\right)=4\left(\frac{6}{4}\right)$$

$$\cancel{4}\left(\frac{3}{\cancel{4}}x\right)=\cancel{4}\left(\frac{6}{\cancel{4}}\right)$$

$$3x=6$$ Solve the equation.

$$\frac{3x}{3}=\frac{6}{3}$$

$$\frac{\cancel{3}x}{\cancel{3}}=\frac{\overset{2}{\cancel{6}}}{\cancel{3}}$$

$$\boxed{x=2}$$

$$\boxed{l_1\cap l_2@(2,2)}$$

Concept Problem Revisited

Solve by elimination:

$$\begin{Bmatrix} 2x+3y=5 \\ 3x-3y=10 \end{Bmatrix}$$

Both equations have a term that is $3y$. In the first equation the coefficient of 'y' is a positive three and in the second equation the coefficient of 'y' is a negative three. If the two equations are added, the 'y' variable is eliminated.

$$\begin{aligned} 2x+3y&=5 \\ \underline{3x-3y}&\underline{=10} \\ 5x&=15 \end{aligned}$$

The resulting equation now has one variable. Solve this equation:

$$\begin{aligned}5x &= 15\\ \frac{5x}{5} &= \frac{15}{5}\\ \frac{\cancel{5}x}{\cancel{5}} &= \frac{\cancelto{3}{15}}{\cancel{5}}\\ &\boxed{x=3}\end{aligned}$$

The value of 'x' is 3. This value can now be substituted into one of the original equations to determine the value of 'y'.

$$\begin{aligned}&2x+3y=5 &&\\ &2(3)+3y=5 && \text{Substitute in the value for } x.\\ &6+3y=5 && \text{Multiply the value of } x \text{ by the coefficient } (2).\\ &6-6+3y=5-6 && \text{Isolate the variable } y.\\ &3y=-1 && \text{Solve the equation.}\\ &\frac{3y}{3}=\frac{-1}{3} &&\\ &\frac{\cancel{3}y}{\cancel{3}}=-\frac{1}{3} &&\\ &\boxed{y=-\frac{1}{3}} &&\end{aligned}$$

The solution to the system of linear equations is $x = 3$ and $y = -\frac{1}{3}$. This solution means

$$\boxed{l_1 \cap l_2 @ \left(3, -\frac{1}{3}\right)}$$

Vocabulary

Elimination Method

The ***elimination method*** is a method used for solving a system of linear equations algebraically. This method involves obtaining an equivalent system of equations such that, when two of the equations are added or subtracted, one of the variables is eliminated.

Guided Practice

1. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix}4x-15y=5\\ 6x-5y=4\end{Bmatrix}$$

2. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} 3x = 7y + 41 \\ 5x = 3y + 51 \end{Bmatrix}.$$

3. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} \frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\ -\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4} \end{Bmatrix}$$

Answers:

1.

$$\begin{Bmatrix} 4x - 15y = 5 \\ 6x - 5y = 4 \end{Bmatrix}$$

Multiply the second equation by (–3) to eliminate the variable 'y'.

$$\begin{aligned} &-3(6x - 5y = 4) \\ &-18x + 15y = -12 \\ &-18x + 15y = -12 \end{aligned}$$

Add the equations:

$$\begin{array}{r} 4x - \cancel{15y} = 5 \\ -18x + \cancel{15y} = -12 \\ \hline -14x = -7 \end{array}$$

Solve the equation:

$$\begin{aligned} &-14x = -7 \\ &\frac{-14x}{-14} = \frac{-7}{-14} \\ &\frac{\cancel{-14}x}{\cancel{-14}} = \frac{-7}{-14} \\ &\boxed{x = \frac{1}{2}} \end{aligned}$$

Substitute this value for 'x' into one of the original equations.

$4x - 15y = 5$

$4\left(\frac{1}{2}\right) - 15y = 5$ Substitute in the value for x.

$2 - 15y = 5$ Multiply the value of x by the coefficient (4).

$2 - 2 - 15y = 5 - 2$ Isolate the variable y.

$-15y = 3$ Solve the equation.

$$\frac{-15y}{-15} = \frac{3}{-15}$$

$$\frac{\cancel{-15}y}{\cancel{-15}} = \frac{3}{-15}$$

$$\boxed{y = -\frac{1}{5}}$$

$$\boxed{l_1 \cap l_2 @ \left(\frac{1}{2}, -\frac{1}{5}\right)}$$

2.

$$\begin{Bmatrix} 3x = 7y + 41 \\ 5x = 3y + 51 \end{Bmatrix}$$

Arrange the equations so that they are of the form

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{Bmatrix}$$

.

$$\begin{aligned} 3x &= 7y + 41 \\ 3x - 7y &= 7y - 7y + 41 \\ 3x - 7y &= 41 \end{aligned} \qquad \begin{aligned} 5x &= 3y + 51 \\ 5x - 3y &= 3y - 3y + 51 \\ 5x - 3y &= 51 \end{aligned}$$

Multiply the first equation by (–5) and the second equation by (3).

$$\begin{aligned} &-5(3x - 7y = 41) \\ &-15x + 35y = -205 \\ &-15x + 35y = -205 \end{aligned} \qquad \begin{aligned} &3(5x - 3y = 51) \\ &15x - 9y = 153 \\ &15x - 9y = 153 \end{aligned}$$

Add the equations to eliminate ’x’.

$$\begin{aligned} -\cancel{15x} + 35y &= -205 \\ \cancel{15x} - 9y &= 153 \\ \hline 26y &= -52 \end{aligned}$$

Solve the equation:

$$26y = 52$$
$$\frac{26y}{26} = \frac{-52}{26}$$
$$\frac{\cancel{26}y}{\cancel{26}} = \frac{\overset{-2}{\cancel{-52}}}{\cancel{26}}$$
$$\boxed{y = -2}$$
$$5x - 3y = 51$$

$5x - 3(-2) = 51$	Substitute in the value for y.
$5x + 6 = 51$	Multiply the value of y by the coefficient (-3).
$5x + 6 - 6 = 51 - 6$	Isolate the variable x.
$5x = 45$	Solve the equation.

$$\frac{5x}{5} = \frac{45}{5}$$
$$\frac{\cancel{5}x}{\cancel{5}} = \frac{\overset{9}{\cancel{45}}}{\cancel{5}}$$
$$\boxed{x = 9}$$
$$\boxed{l_1 \cap l_2 @ (9, -2)}$$

3.

$$\left\{\begin{array}{c} \frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\ -\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4} \end{array}\right\}$$

Begin by multiplying each equation by the LCM of the denominators to simplify the system.
$\frac{2}{5}m + \frac{3}{4}n = \frac{5}{2}$ The LCM for the denominators is 20.

$$20\left(\frac{2}{5}\right)m + 20\left(\frac{3}{4}\right)n = 20\left(\frac{5}{2}\right)$$
$$\overset{4}{\cancel{20}}\left(\frac{2}{\cancel{5}}\right)m + \overset{5}{\cancel{20}}\left(\frac{3}{\cancel{4}}\right)n = \overset{10}{\cancel{20}}\left(\frac{5}{\cancel{2}}\right)$$
$$8m + 15n = 50$$
$$\boxed{8m + 15n = 50}$$

$$-\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4}$$ The LCM for the denominators is 12.

$$-12\left(\frac{2}{3}\right)m + 12\left(\frac{1}{2}\right)n = 12\left(\frac{3}{4}\right)$$
$$-\overset{4}{\cancel{12}}\left(\frac{2}{\cancel{3}}\right)m + \overset{6}{\cancel{12}}\left(\frac{1}{\cancel{2}}\right)n = \overset{3}{\cancel{12}}\left(\frac{3}{\cancel{4}}\right)$$
$$-8m + 6n = 9$$
$$\boxed{-8m + 6n = 9}$$

The two equations that need to be solved are:

$$\left\{\begin{array}{c} 8m + 15n = 50 \\ -8m + 6n = 9 \end{array}\right\}$$

The equations will be solved by using the elimination method. The variable 'm' has the same numerical coefficient with opposite signs. The variable will be eliminated when the equations are added.

$$\begin{array}{r} \cancel{8m}+15n=50 \\ \underline{-\cancel{8m}+6n=9} \\ 21n=59 \end{array}$$

Solve the equation:

$$21n=59$$

$$\frac{21n}{21}=\frac{59}{21}$$

$$\frac{\cancel{21}n}{\cancel{21}}=\frac{59}{21}$$

$$\boxed{n=\frac{59}{21}}$$

$$\frac{2}{5}m+\frac{3}{4}n=\frac{5}{2}$$

$$\frac{2}{5}m+\frac{3}{4}\left(\frac{59}{21}\right)=\frac{5}{2}$$ Substitute in the value for n.

$$\frac{2}{5}m+\frac{177}{84}=\frac{5}{2}$$ Multiply the value of y by the coefficient $\left(\frac{59}{21}\right)$.

$$\frac{2}{5}m+\frac{177}{84}-\frac{177}{84}=\frac{5}{2}-\frac{177}{84}$$ Isolate the variable x.

$$\frac{2}{5}m=\frac{210}{84}-\frac{177}{84}$$

$$\frac{2}{5}m=\frac{33}{84}$$ Solve the equation.

$$420\left(\frac{2}{5}\right)x=420\left(\frac{33}{84}\right)$$

$$\overset{84}{\cancel{420}}\left(\frac{2}{\cancel{5}}\right)x=\overset{5}{\cancel{420}}\left(\frac{33}{\cancel{84}}\right)$$

$$168x=165$$

$$\frac{168x}{168}=\frac{165}{168}$$

$$\frac{\cancel{168}x}{\cancel{168}}=\frac{165}{168}$$

$$\boxed{x=\frac{55}{56}}$$

$$\boxed{l_1\cap l_2 @\left(\frac{55}{56},\frac{59}{21}\right)}$$

Practice

Solve the following systems of linear equations using the elimination method.

1.

$$\begin{Bmatrix} 16x - y - 181 = 0 \\ 19x - y = 214 \end{Bmatrix}$$

2.

$$\begin{Bmatrix} 3x + 2y + 9 = 0 \\ 4x = 3y + 5 \end{Bmatrix}$$

3.

$$\begin{Bmatrix} x = 7y + 38 \\ 14y = -x - 46 \end{Bmatrix}$$

4.

$$\begin{Bmatrix} 2x + 9y = -1 \\ 4x + y = 15 \end{Bmatrix}$$

5.

$$\begin{Bmatrix} x - \frac{3}{5}y = \frac{26}{5} \\ 4y = 61 - 7x \end{Bmatrix}$$

6.

$$\begin{Bmatrix} 3x - 5y = 12 \\ 2x + 10y = 4 \end{Bmatrix}$$

7.

$$\begin{Bmatrix} 3x + 2y + 9 = 0 \\ 4x = 3y + 5 \end{Bmatrix}$$

8.

$$\begin{Bmatrix} x = 69 + 6y \\ 3x = 4y - 45 \end{Bmatrix}$$

9.

$$\begin{Bmatrix} 3(x-1)-4(y+2)=-5 \\ 4(x+5)-(y-1)=16 \end{Bmatrix}$$

10.

$$\begin{Bmatrix} \frac{3}{4}x-\frac{2}{5}y=2 \\ \frac{1}{7}x+\frac{3}{2}y=\frac{113}{7} \end{Bmatrix}$$

11.

$$\begin{Bmatrix} 3x+5y=17 \\ 2x+3y=11 \end{Bmatrix}$$

12.

$$\begin{Bmatrix} 3x-5y=-29 \\ 2x-8y=-42 \end{Bmatrix}$$

13.

$$\begin{Bmatrix} 7x-8y=-26 \\ 5x-12y=-45 \end{Bmatrix}$$

14.

$$\begin{Bmatrix} 6x+5y=5.1 \\ 4x-2y=-1.8 \end{Bmatrix}$$

15. When does it make sense to use the elimination method to solve a system of equations?

5.4 Consistent and Inconsistent Linear Systems

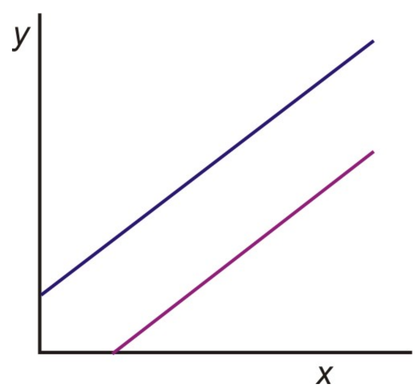
Here you'll learn the difference between three special types of linear systems: **inconsistent linear systems**, **consistent linear systems**, and **dependent linear systems**. You'll then use that information to determine the number of solutions a system has.

What if you were given a system of equations like $2x - y = 5$ and $10x - 5y = 25$? How could you rewrite these equations to determine the number of solutions the system has? After completing this Concept, you'll be able to identify whether a system of equations like this one is an inconsistent one, a consistent one, or a dependent one.

Guidance

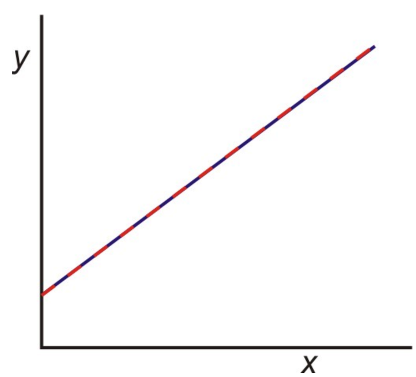
As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Or at least that's what usually happens. But what if the lines turn out to be parallel when we graph them?



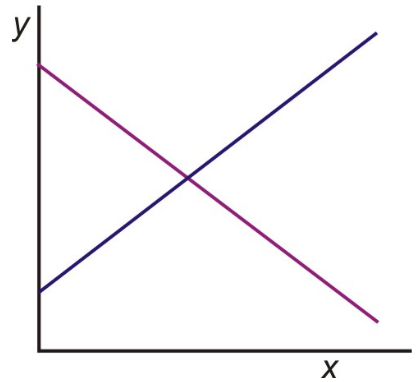
If the lines are parallel, they won't ever intersect. That means that the system of equations they represent has no solution. A system with no solutions is called an **inconsistent system.**

And what if the lines turn out to be identical?



If the two lines are the same, then *every* point on one line is also on the other line, so every point on the line is a solution to the system. The system has an **infinite number** of solutions, and the two equations are really just different forms of the same equation. Such a system is called a **dependent system**.

But usually, two lines cross at exactly one point and the system has exactly one solution:



A system with exactly one solution is called a **consistent system**.

To identify a system as **consistent, inconsistent**, or **dependent**, we can graph the two lines on the same graph and see if they intersect, are parallel, or are the same line. But sometimes it is hard to tell whether two lines are parallel just by looking at a roughly sketched graph.

Another option is to write each line in slope-intercept form and compare the slopes and $y-$ intercepts of the two lines. To do this we must remember that:

- Lines with different slopes always intersect.
- Lines with the same slope but different $y-$intercepts are parallel.
- Lines with the same slope and the same $y-$intercepts are identical.

Example A

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned}2x-5y&=2\\4x+y&=5\end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$2x-5y=2 \Rightarrow -5y=-2x+2 \Rightarrow y=\frac{2}{5}x-\frac{2}{5}$

$4x+y=5 \Rightarrow y=-4x+5$

The slopes of the two equations are different; therefore the lines must cross at a single point and the system has exactly one solution. This is a **consistent system.**

Example B

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned}3x&=5-4y\\6x+8y&=7\end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$3x = 5 - 4y \Rightarrow 4y = -3x + 5 \Rightarrow y = -\frac{3}{4}x + \frac{5}{4}$

$6x + 8y = 7 \Rightarrow 8y = -6x + 7 \Rightarrow y = -\frac{3}{4}x + \frac{7}{8}$

The slopes of the two equations are the same but the $y-$intercepts are different; therefore the lines are parallel and the system has no solutions. This is an **inconsistent system.**

Example C

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x + y &= 3 \\ 3x + 3y &= 9 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$x + y = 3 \Rightarrow y = -x + 3$

$3x + 3y = 9 \Rightarrow 3y = -3x + 9 \Rightarrow y = -x + 3$

The lines are identical; therefore the system has an infinite number of solutions. It is a **dependent system.**

Vocabulary

- A system with no solutions is called an **inconsistent system.** For linear equations, this occurs with parallel lines.
- A system where the two equations overlap at one, multiple, or infinitely many points is called a **consistent system.**
- **Coincident lines** are lines with the same slope and $y-$intercept. The lines completely overlap.
- When solving a system of coincident lines, the resulting equation will be without variables and the statement will be true. You can conclude the system has an infinite number of solutions. This is called a **consistent-dependent system**.

Guided Practice

Determine whether the following system of linear equations has zero, one, or infinitely many solutions:

$$\begin{cases} 2y + 6x = 20 \\ y = -3x + 7 \end{cases}$$

What kind of system is this?

Solution:

It is easier to compare equations when they are in the same form. We will rewrite the first equation in slope-intercept form.

$2y+6x=20 \Rightarrow y+3x=10 \Rightarrow y=-3x+10$

Since the two equations have the same slope, but different y-intercepts, they are different but parallel lines. Parallel lines never intersect, so they have no solutions.

Since the lines are parallel, it is an inconsistent system.

Practice

Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.

1. $$\begin{aligned} 3x-4y&=13 \\ y&=-3x-7 \end{aligned}$$

2. $$\begin{aligned} \frac{3}{5}x+y&=3 \\ 1.2x+2y&=6 \end{aligned}$$

3. $$\begin{aligned} 3x-4y&=13 \\ y&=-3x-7 \end{aligned}$$

4. $$\begin{aligned} 3x-3y&=3 \\ x-y&=1 \end{aligned}$$

5. $$\begin{aligned} 0.5x-y&=30 \\ 0.5x-y&=-30 \end{aligned}$$

6. $$\begin{aligned} 4x-2y&=-2 \\ 3x+2y&=-12 \end{aligned}$$

7. $$\begin{aligned} 3x+y&=4 \\ y&=5-3x \end{aligned}$$

8. $$\begin{aligned} x-2y&=7 \\ 4y-2x&=14 \end{aligned}$$

9.

$$\begin{aligned}-2y+4x&=8\\ y-2x&=-4\end{aligned}$$

10.

$$\begin{aligned}x-\frac{y}{2}&=\frac{3}{2}\\ 3x+y&=6\end{aligned}$$

11.

$$\begin{aligned}0.05x+0.25y&=6\\ x+y&=24\end{aligned}$$

12.

$$\begin{aligned}x+\frac{2y}{3}&=6\\ 3x+2y&=2\end{aligned}$$

5.5 Systems of Linear Inequalities

Here you'll learn how to graph and solve a system of two or more linear inequalities. You'll also determine if such systems are consistent or inconsistent.

What if you were given a system of linear inequalities like $6x-2y \geq 3$ and $2y-3x \leq 7$? How could you determine its solution? After completing this Concept, you'll be able to find the solution region of systems of linear inequalities like this one.

Guidance

In the last chapter you learned how to graph a linear inequality in two variables. To do that, you graphed the equation of the straight line on the coordinate plane. The line was solid for $\leq$ or $\geq$ signs (where the equals sign is included), and the line was dashed for $<$or $>$signs (where the equals sign is not included). Then you shaded above the line (if the inequality began with $y >$ or $y \geq$) or below the line (if it began with $y <$ or $y \leq$).

In this section, we'll see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes.

Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example A

Solve the following system:

$$\begin{aligned} 2x+3y &\leq 18 \\ x-4y &\leq 12 \end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

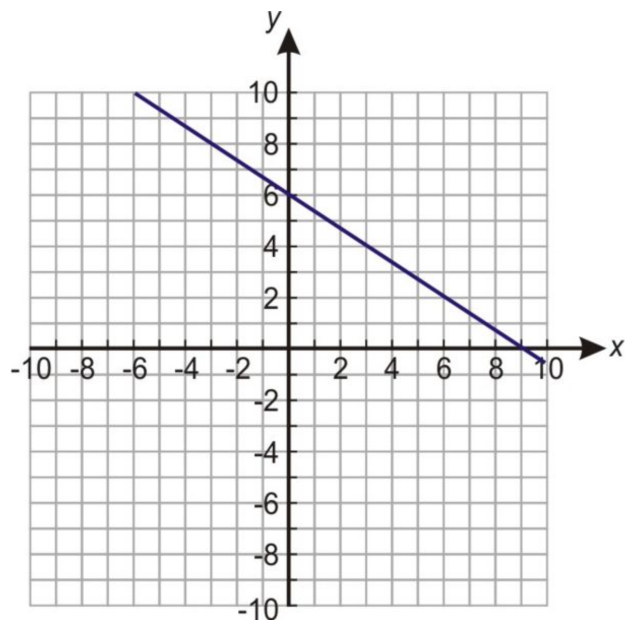
First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

$$\begin{aligned} 3y &\leq -2x+18 & \qquad y &\leq -\frac{2}{3}x+6 \\ & & &\Rightarrow \\ -4y &\leq -x+12 & \qquad y &\geq \frac{x}{4}-3 \end{aligned}$$

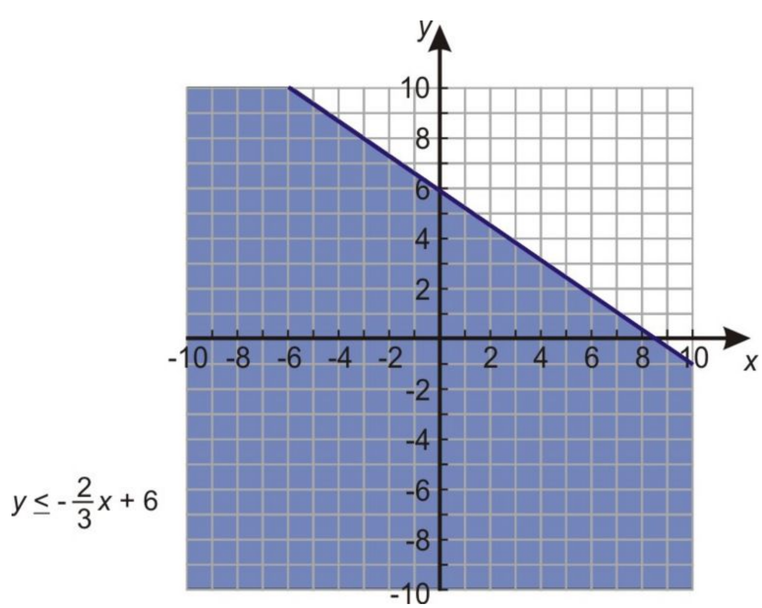
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

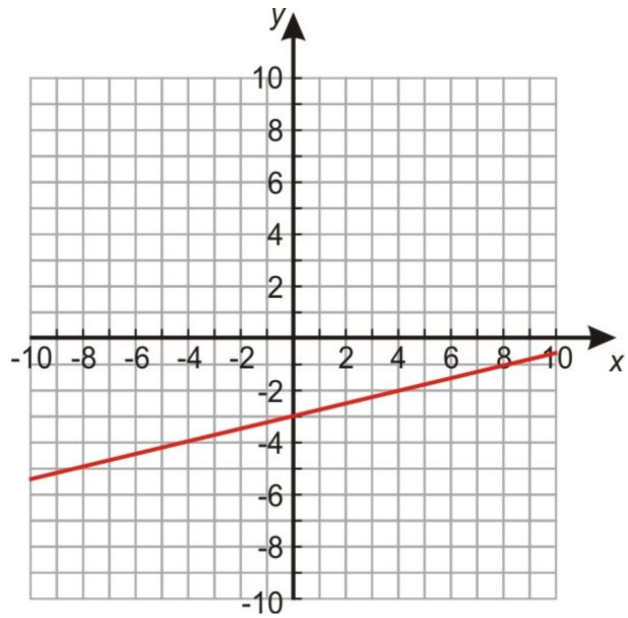
Here's the graph of the first inequality:



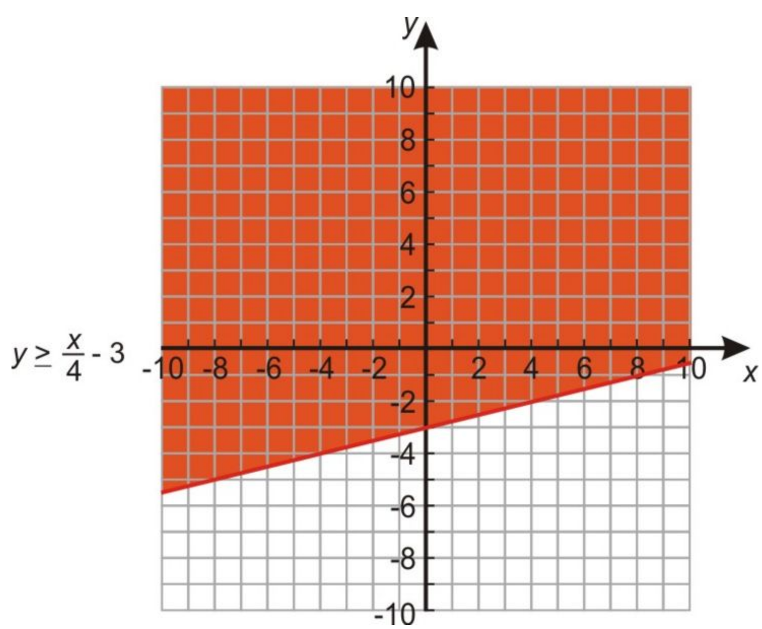
The line is solid because the equals sign is included in the inequality. Since the inequality is **less** than or equal to, we shade **below** the line.



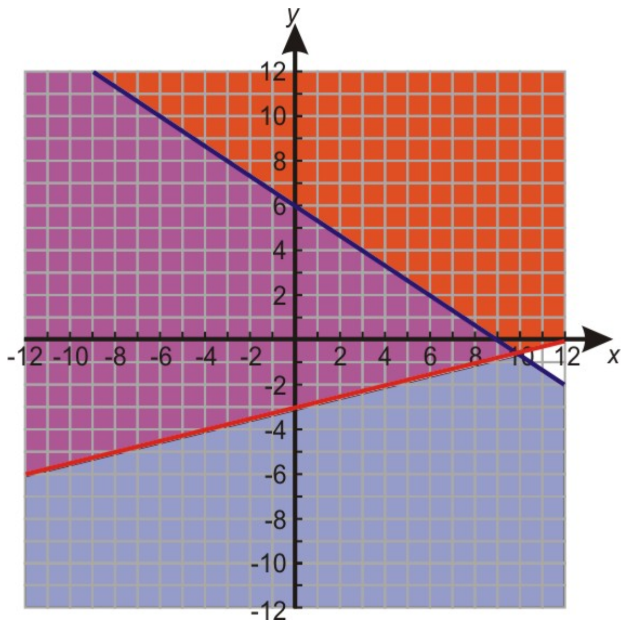
And here's the graph of the second inequality:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because y is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

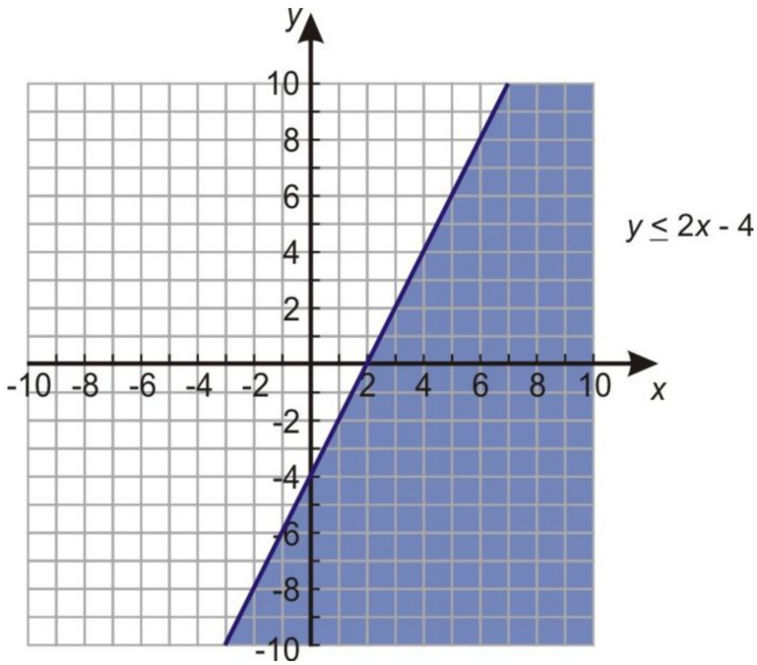
Example B

There are also situations where a system of inequalities has no solution. For example, let's solve this system.

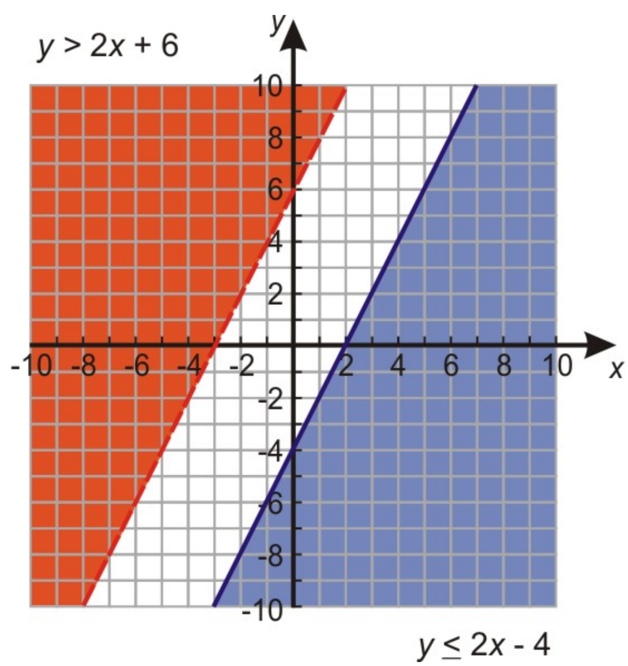
$$y \leq 2x - 4$$
$$y > 2x + 6$$

Solution

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because y is greater than.



It doesn't look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won't ever overlap since the lines won't ever cross. So this system of inequalities has no solution.

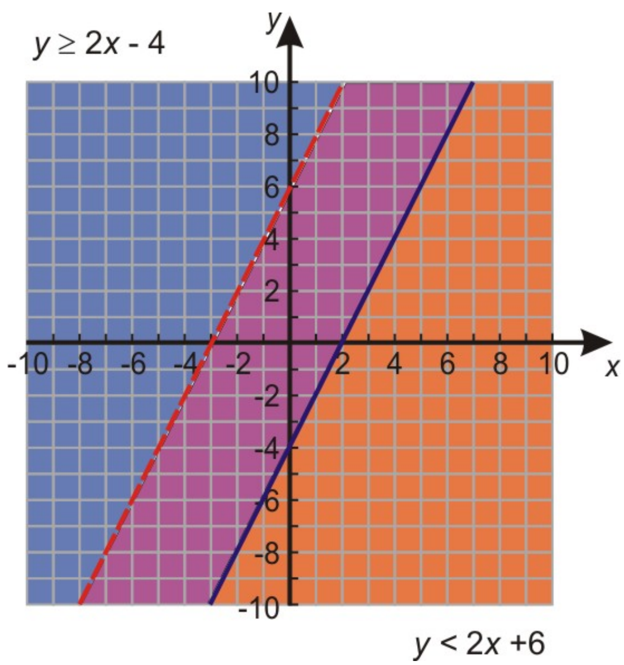
But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

To graph the system

$$y \geq 2x - 4$$
$$y < 2x + 6$$

,

we draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region—one that continues infinitely in at least one direction. But if we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded**—a finite region with three or more sides.

Let's look at a simple example.

Example C

Find the solution to the following system of inequalities.

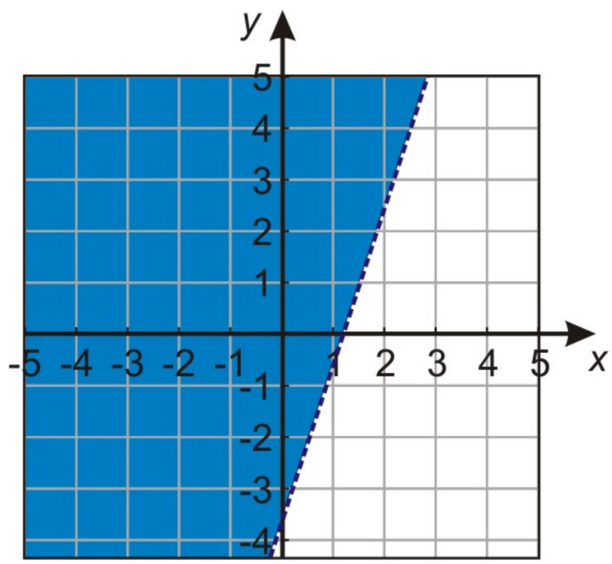
$$3x - y < 4$$
$$4y + 9x < 8$$
$$x \geq 0$$
$$y \geq 0$$

Solution

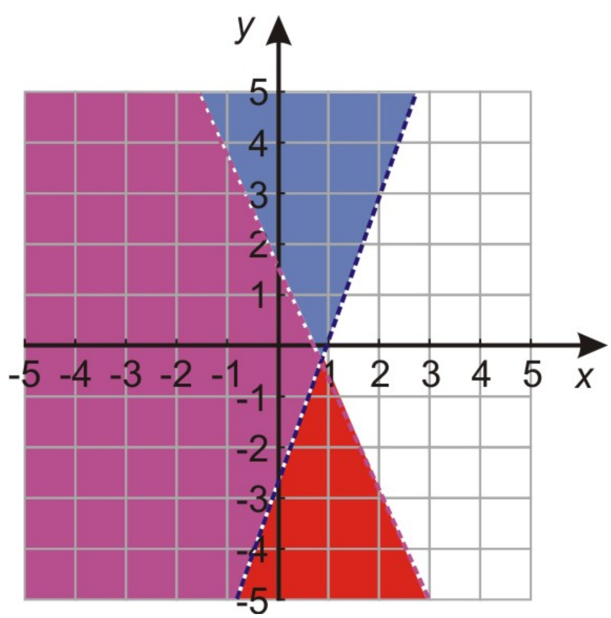
Let's start by writing our inequalities in slope-intercept form.

$$y > 3x - 4$$
$$y < -\frac{9}{4}x + 2$$
$$x \geq 0$$
$$y \geq 0$$

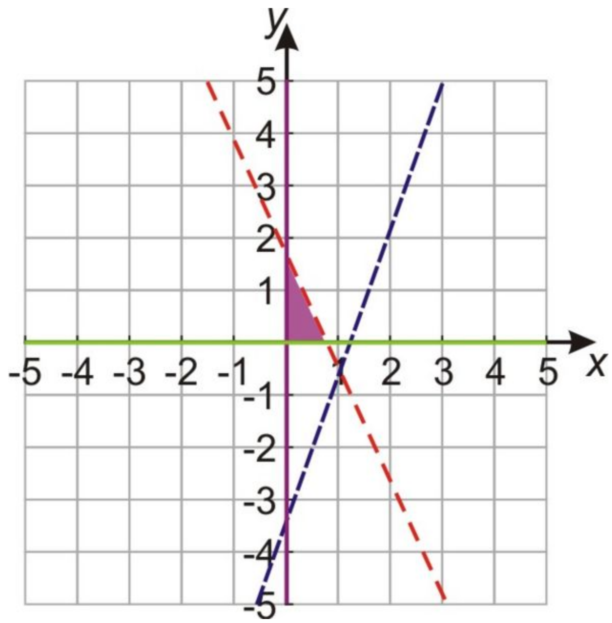
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$:



Next we graph $y < -\frac{9}{4}x + 2$:



Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.



The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x - 4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

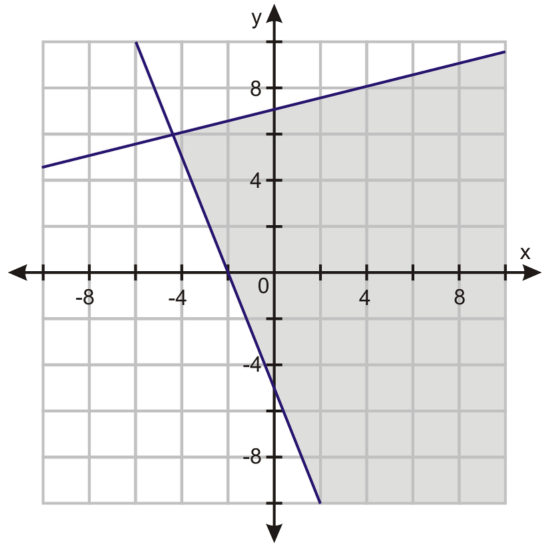
But that wasn't obvious until we actually drew the graph!

Vocabulary

- **Solution for the system of inequalities:** The ***solution for the system of inequalities*** is the common shaded region between all the inequalities in the system.
- **Feasible region:** The common shaded region of the system of inequalities is called the ***feasible region***.
- **Optimization:** The goal is to locate the feasible region of the system and use it to answer a profitability, or ***optimization***, question.

Guided Practice

Write the system of inequalities shown below.



Solution:

There are two boundary lines, so there are two inequalities. Write each one in slope-intercept form.

$$y \leq \frac{1}{4}x + 7$$
$$y \geq -\frac{5}{2}x - 5$$

Practice

1. Consider the system

$$y < 3x - 5$$
$$y > 3x - 5$$

. Is it consistent or inconsistent? Why?

2. Consider the system

$$y \leq 2x + 3$$
$$y \geq 2x + 3$$

. Is it consistent or inconsistent? Why?

3. Consider the system

$$y \leq -x + 1$$
$$y > -x + 1$$

. Is it consistent or inconsistent? Why?

4. In example 3 in this lesson, we solved a system of four inequalities and saw that one of the inequalities, $y > 3x - 4$, didn't affect the solution set of the system.
    a. What would happen if we changed that inequality to $y < 3x - 4$?
    b. What's another inequality that we could add to the original system without changing it? Show how by sketching a graph of that inequality along with the rest of the system.

c. What's another inequality that we could add to the original system to make it inconsistent? Show how by sketching a graph of that inequality along with the rest of the system.

5. Recall the compound inequalities in one variable that we worked with back in chapter 6. Compound inequalities with "and" are simply systems like the ones we are working with here, except with one variable instead of two.
   a. Graph the inequality $x > 3$ in two dimensions. What's another inequality that could be combined with it to make an inconsistent system?
   b. Graph the inequality $x \leq 4$ on a number line. What two-dimensional system would have a graph that looks just like this one?

Find the solution region of the following systems of inequalities.

6.

$$\begin{aligned} x - y &< -6 \\ 2y &\geq 3x + 17 \end{aligned}$$

7.

$$\begin{aligned} 4y - 5x &< 8 \\ -5x &\geq 16 - 8y \end{aligned}$$

8.

$$\begin{aligned} 5x - y &\geq 5 \\ 2y - x &\geq -10 \end{aligned}$$

9.

$$\begin{aligned} 5x + 2y &\geq -25 \\ 3x - 2y &\leq 17 \\ x - 6y &\geq 27 \end{aligned}$$

10.

$$\begin{aligned} 2x - 3y &\leq 21 \\ x + 4y &\leq 6 \\ 3x + y &\geq -4 \end{aligned}$$

11.

$$\begin{aligned} 12x - 7y &< 120 \\ 7x - 8y &\geq 36 \\ 5x + y &\geq 12 \end{aligned}$$

5.6 Linear Programming

Here you'll learn how to analyze and find the feasible solution(s) to a system of inequalities under a given set of constraints.

What if you had an equation like $z = x + y$ in which a set of contraints like $x - y \leq 4$, $x + y \leq 2$, and $2x + 3y \geq -3$ were placed on it. How could you find the minimum and maximum values of z? After completing this Concept, you'll be able to analyze a system of inequalities to make the best decisions given the constraints of the situation.

Guidance

A lot of interesting real-world problems can be solved with systems of linear inequalities.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend only waits tables in a certain region of the restaurant. The restaurant is also known for its great views, so you want to sit in a certain area of the restaurant that offers a good view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best view and be served by your friend.

Often, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the *best* possible value under those conditions. A typical example would be taking the limitations of materials and labor at a factory, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real-life systems can have dozens or hundreds of variables, or more. In this section, we'll only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the coordinate plane (called **the feasibility region**).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the system of equations that applies to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

Example A

If $z = 2x + 5y$, find the maximum and minimum values of z given these constraints:

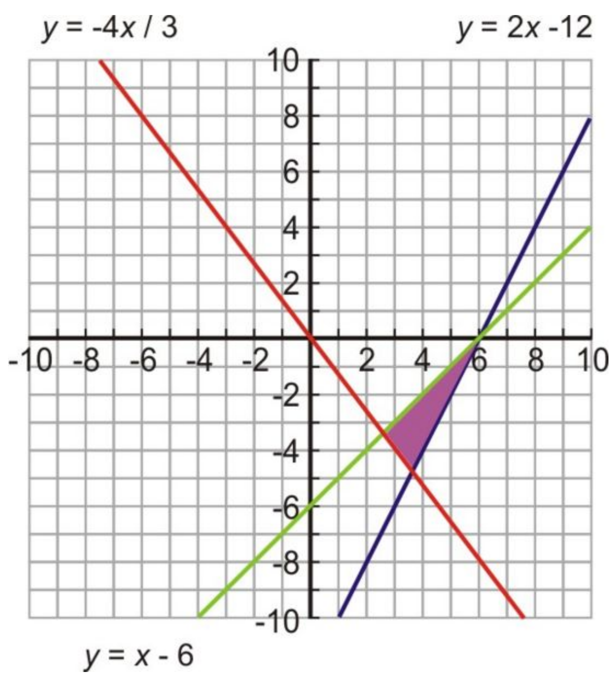
$$\begin{aligned} 2x - y &\leq 12 \\ 4x + 3y &\geq 0 \\ x - y &\geq 6 \end{aligned}$$

Solution

First, we need to find the solution to this system of linear inequalities by graphing and shading appropriately. To graph the inequalities, we rewrite them in slope-intercept form:

$$y \geq 2x - 12$$
$$y \geq -\frac{4}{3}x$$
$$y \leq x - 6$$

These three linear inequalities are called the **constraints**, and here is their graph:



The shaded region in the graph is called the **feasibility region**. All possible solutions to the system occur in that region; now we must try to find the maximum and minimum values of the variable z within that region. In other words, which values of x and y within the feasibility region will give us the greatest and smallest overall values for the expression $2x+5y$?

Fortunately, we don't have to test every point in the region to find that out. It just so happens that the minimum or maximum value of the optimization equation in a linear system like this will always be found at one of the vertices (the corners) of the feasibility region; we just have to figure out *which* vertices. So for each vertex—each point where two of the lines on the graph cross—we need to solve the system of just those two equations, and then find the value of z at that point.

The first system consists of the equations $y = 2x - 12$ and $y = -\frac{4}{3}x$. We can solve this system by substitution:

$$-\frac{4}{3}x = 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6$$
$$y = 2x - 12 \Rightarrow y = 2(3.6) - 12 \Rightarrow y = -4.8$$

The lines intersect at the point (3.6, -4.8).

The second system consists of the equations $y = 2x - 12$ and $y = x - 6$. Solving this system by substitution:

$$x - 6 = 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6$$
$$y = x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 0$$

The lines intersect at the point (6, 0).

The third system consists of the equations $y = -\frac{4}{3}x$ and $y = x - 6$. Solving this system by substitution:

$$x - 6 = -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57$$
$$y = x - 6 \Rightarrow y = 2.57 - 6 \Rightarrow y = -3.43$$

The lines intersect at the point (2.57, -3.43).

So now we have three different points that might give us the maximum and minimum values for z. To find out which ones actually do give the maximum and minimum values, we can plug the points into the optimization equation $z = 2x + 5y$.

When we plug in (3.6, -4.8), we get $z = 2(3.6) + 5(-4.8) = -16.8$.

When we plug in (6, 0), we get $z = 2(6) + 5(0) = 12$.

When we plug in (2.57, -3.43), we get $z = 2(2.57) + 5(-3.43) = -12.01$.

So we can see that **the point (6, 0) gives us the maximum possible value for z and the point (3.6, –4.8) gives us the minimum value.**

In the previous example, we learned how to apply the method of linear programming in the abstract. In the next example, we'll look at a real-life application.

Example B

You have $10,000 to invest, and three different funds to choose from. The municipal bond fund has a 5% return, the local bank's CDs have a 7% return, and a high-risk account has an expected 10% return. To minimize risk, you decide not to invest any more than $1,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. What's the best way to distribute your money given these constraints?

Solution:

Let's define our variables:

x is the amount of money invested in the municipal bond at 5% return

y is the amount of money invested in the bank's CD at 7% return

$10000 - x - y$ is the amount of money invested in the high-risk account at 10% return

z is the total interest returned from all the investments, so $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of z.

Now, let's write inequalities for the *constraints*:

You decide not to invest more than $1000 in the high-risk account—that means:

$$10000 - x - y \leq 1000$$

You need to invest at least three times as much in the municipal bonds as in the bank CDs—that means:

$$3y \leq x$$

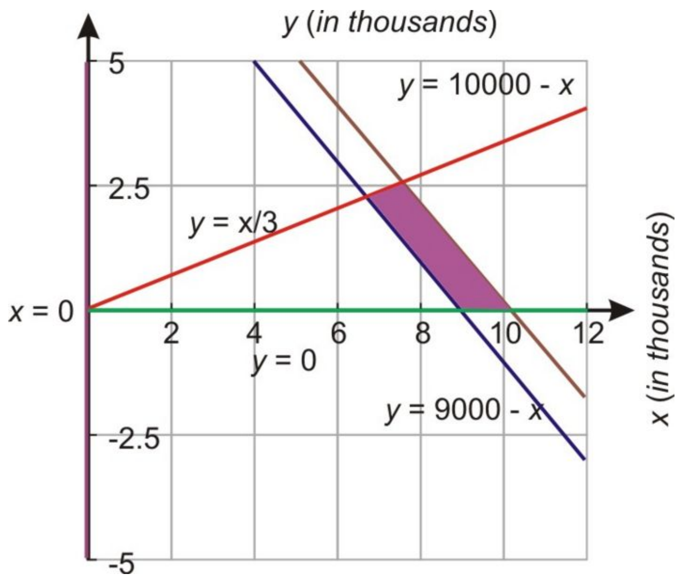
Also, you can't invest less than zero dollars in each account, so:

$$x \geq 0$$
$$y \geq 0$$
$$10000 - x - y \geq 0$$

To summarize, we must maximize the expression $z = 1000 - .05x - .03y$ using the constraints:

$10000 - x - y \leq 1000$		$y \geq 9000 - x$
$3y \leq x$		$y \leq \frac{x}{3}$
$x \geq 0$	Or in slope-intercept form:	$x \geq 0$
$y \geq 0$		$y \geq 0$
$10000 - x - y \geq 0$		$y \leq 10000 - x$

Step 1: Find the solution region to the set of inequalities by graphing each line and shading appropriately. The following figure shows the overlapping region:



The purple region is the feasibility region where all the possible solutions can occur.

Step 2: Next we need to find the corner points of the feasibility region. Notice that there are four corners. To find their coordinates, we must pair up the relevant equations and solve each resulting system.

System 1:

$$y = \frac{x}{3}$$
$$y = 10000 - x$$

Substitute the first equation into the second equation:

$$\frac{x}{3} = 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow 4x = 30000 \Rightarrow x = 7500$$
$$y = \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500$$

The intersection point is (7500, 2500).

System 2:

$$y = \frac{x}{3}$$
$$y = 9000 - x$$

Substitute the first equation into the second equation:

$$\frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750$$
$$y = \frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250$$

The intersection point is (6750, 2250).

System 3:

$$y = 0$$
$$y = 10000 - x$$

The intersection point is (10000, 0).

System 4:

$$y = 0$$
$$y = 9000 - x$$

The intersection point is (9000, 0).

Step 3: In order to find the maximum value for z, we need to plug all the intersection points into the equation for z and find which one yields the largest number.

(7500, 2500): $z = 1000 - 0.05(7500) - 0.03(2500) = 550$

(6750, 2250): $z = 1000 - 0.05(6750) - 0.03(2250) = 595$

(10000, 0): $z = 1000 - 0.05(10000) - 0.03(0) = 500$

(9000, 0): $z = 1000 - 0.05(9000) - 0.03(0) = 550$

The maximum return on the investment of \$595 occurs at the point (6750, 2250). This means that:

\$6,750 is invested in the municipal bonds.

\$2,250 is invested in the bank CDs.

\$1,000 is invested in the high-risk account.

Example C

James is trying to expand his pastry business to include cupcakes and personal cakes. He has 40 hours available to decorate the new items and can use no more than 22 pounds of cake mix. Each personal cake requires 2 pounds of cake mix and 2 hours to decorate. Each cupcake order requires one pound of cake mix and 4 hours to decorate. If he can sell each personal cake for \$14.99 and each cupcake order for \$16.99, how many personal cakes and cupcake orders should James make to make the most revenue?

There are four inequalities in this situation. First, state the variables. Let $p = \textit{the number of personal cakes}$ and $c = \textit{the number of cupcake orders}$.

Translate this into a system of inequalities.

$2p+1c \le 22$ –This is the amount of available cake mix.

$2p+4c \le 40$ –This is the available time to decorate.

$p \ge 0$ –You cannot make negative personal cakes.

$c \ge 0$ –You cannot make negative cupcake orders.

Now graph each inequality and determine the feasible region.

The feasible region has four vertices: $\{(0,\ 0),(0,\ 10),(11,\ 0),(8,\ 6)\}$. According to our theorem, the optimization answer will **only occur** at one of these vertices.

Write the optimization equation. How much of each type of order should James make to bring in the most revenue?

$$14.99p+16.99c = \textit{maximum revenue}$$

Substitute each ordered pair to determine which makes the most money.

$$\begin{aligned}
(0,0) &\rightarrow \$0.00 \\
(0,10) &\rightarrow 14.99(0)+16.99(10)=\$169.90 \\
(11,0) &\rightarrow 14.99(11)+16.99(0)=\$164.89 \\
(8,6) &\rightarrow 14.99(8)+16.99(6)=\$221.86
\end{aligned}$$

To make the most revenue, James should make 8 personal cakes and 6 cupcake orders.

Vocabulary

- **Linear programming** is the mathematical process of analyzing a system of inequalities to make the best decisions given the constraints of the situation.
- **Constraints** are the particular restrictions of a situation due to time, money, or materials.
- In an **optimization** problem, the goal is to locate the feasible region of the system and use it to answer a profitability, or ***optimization***, question.
- The **solution for the system of inequalities** is the common shaded region between all the inequalities in the system.
- The common shaded region of the system of inequalities is called the **feasible region**.

Guided Practice

Graph the solution to the following system:

$$\begin{aligned}
x-y &< -6 \\
2y &\geq 3x+17
\end{aligned}$$

Solution:

First we will rewrite the equations in slope-intercept form in order to graph them:

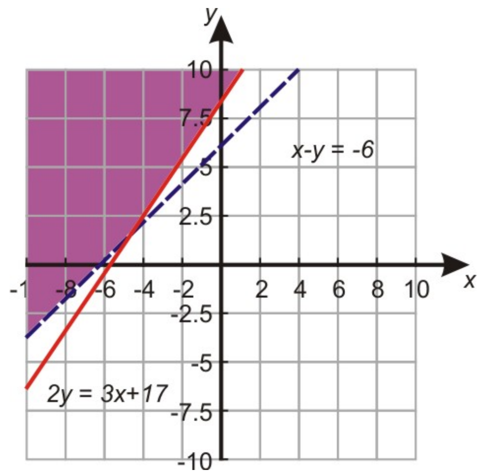
Inequality 1

$$\begin{aligned}
x-y &< -6 && \text{Solve for y.} \\
-y &< -x-6 && \text{Subtract x from each side.} \\
y &> x+6 && \text{Multiply each side by -1, flipping the inequality.}
\end{aligned}$$

Inequality 2

$$\begin{aligned}
2y &\geq 3x+17 < && \text{Solve for y.} \\
y &\geq \frac{3}{2}x+8.5 < && \text{Divide each side by 2.}
\end{aligned}$$

Graph each equation and shade accordingly:



Practice

Solve the following linear programming problems.

1. Given the following constraints, find the maximum and minimum values of $z = -x + 5y$:

$$\begin{aligned} x + 3y &\leq 0 \\ x - y &\geq 0 \\ 3x - 7y &\leq 16 \end{aligned}$$

Santa Claus is assigning elves to work an eight-hour shift making toy trucks. Apprentice elves draw a wage of five candy canes per hour worked, but can only make four trucks an hour. Senior elves can make six trucks an hour and are paid eight candy canes per hour. There's only room for nine elves in the truck shop, and due to a candy-makers' strike, Santa Claus can only pay out 480 candy canes for the whole 8-hour shift.

2. How many senior elves and how many apprentice elves should work this shift to maximize the number of trucks that get made?
3. How many trucks will be made?
4. Just before the shift begins, the apprentice elves demand a wage increase; they insist on being paid seven candy canes an hour. Now how many apprentice elves and how many senior elves should Santa assign to this shift?
5. How many trucks will now get made, and how many candy canes will Santa have left over?

In Adrian's Furniture Shop, Adrian assembles both bookcases and TV cabinets. Each type of furniture takes her about the same time to assemble. She figures she has time to make at most 18 pieces of furniture by this Saturday. The materials for each bookcase cost her \$20 and the materials for each TV stand costs her \$45. She has \$600 to spend on materials. Adrian makes a profit of \$60 on each bookcase and a profit of \$100 on each TV stand.

6. Set up a system of inequalities. What $x-$ and $y-$values do you get for the point where Adrian's profit is maximized? Does this solution make sense in the real world?
7. What two possible real-world $x-$values and what two possible real-world $y-$values would be closest to the values in that solution?
8. With two choices each for x and y, there are four possible combinations of $x-$ and $y-$values. Of those four combinations, which ones actually fall within the feasibility region of the problem?
9. Which one of those feasible combinations seems like it would generate the most profit? Test out each one to confirm your guess. How much profit will Adrian make with that combination?

10. Based on Adrian's previous sales figures, she doesn't think she can sell more than 8 TV stands. Now how many of each piece of furniture should she make, and what will her profit be?
11. Suppose Adrian is confident she can sell all the furniture she can make, but she doesn't have room to display more than 7 bookcases in her shop. Now how many of each piece of furniture should she make, and what will her profit be?
12. Here's a "linear programming" problem on a line instead of a plane: Given the constraints $x \leq 5$ and $x \geq -2$, maximize the value of y where $y = x + 3$.