Chapter 6: Exponential Functions

CHAPTER 6

Chapter Outline

6.1 Product Rules for Exponents

Here you'll learn how to multiply two terms with the same base and how to find the power of a product.

Suppose you have the expression:

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$

How could you write this expression in a more concise way?

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and products.

RULE: To multiply two terms with the same base, add the exponents.

$$a^m \times a^n = \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}}$$

$$a^m \times a^n = \underbrace{(a \times a \times a \ldots \times a)}_{m+n \text{ factors}}$$

$$a^m \times a^n = a^{m+n}$$

RULE: To raise a product to a power, raise each of the factors to the power.

$$(ab)^n = \underbrace{(ab) \times (ab) \times \ldots \times (ab)}_{n \text{ factors}}$$

$$(ab)^n = \underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}} \times \underbrace{(b \times b \times \ldots \times b)}_{n \text{ factors}}$$

$$(ab)^n = a^n b^n$$

Example A

Simplify $3^2 \times 3^3$.

Solution:

$3^2 \times 3^3$	The base is 3.
3^{2+3}	Keep the base of 3 and add the exponents.
3^5	This answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3$$
$$3^5 = 243$$
$$\boxed{3^2 \times 3^3 = 3^5 = 243}$$

Example B

Simplify $(x^3)(x^6)$.

Solution:

$(x^3)(x^6)$	The base is x.
x^{3+6}	Keep the base of x and add the exponents.
x^9	The answer is in exponential form.
$\boxed{(x^3)(x^6) = x^9}$	

Example C

Simplify $y^5 \cdot y^2$.

Solution:

$y^5 \cdot y^2$	The base is y.
y^{5+2}	Keep the base of y and add the exponents.
y^7	The answer is in exponential form.
$\boxed{y^5 \cdot y^2 = y^7}$	

Example D

Simplify $5x^2y^3 \cdot 3xy^2$.

Solution:

$5x^2y^3 \cdot 3xy^2$	The bases are x and y.
$15(x^2y^3)(xy^2)$	Multiply the coefficients - $5 \times 3 = 15$. Keep the base of x and y and add the exponents of the same base. If a base does not have a written exponent, it is understood as 1.
$15x^{2+1}y^{3+2}$	
$15x^3y^5$	The answer is in exponential form.
$\boxed{5x^2y^3 \cdot 3xy^2 = 15x^3y^5}$	

Concept Problem Revisited

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$ can be rewritten as $x^9y^5x^4$. Then, you can use the rules of exponents to simplify the expression to $x^{13}y^5$. This is certainly much quicker to write!

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

Simplify each of the following expressions.

1. $(-3x)^2$
2. $(5xy)^3$
3. $(2^3 \cdot 3^2)^2$

Answers:

1. $9x^2$. Here are the steps:

$$\begin{aligned}(-3x)^2 &= (-3)^2 \cdot (x)^2 \\ &= 9x^2\end{aligned}$$

2. $125x^3y^3$. Here are the steps:

$$\begin{aligned}(5x^2y^4)^3 &= (5)^3 \cdot (x)^3 \cdot (y)^3 \\ &= 125x^3y^3\end{aligned}$$

3. 5184. Here are the steps:

$$\begin{aligned}(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2 \\ &= (72)^2 \\ &= 5184\end{aligned}$$

OR

$$\begin{aligned}(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2 \\ &= 8^2 \cdot 9^2 \\ &= 64 \cdot 81 \\ &= 5184\end{aligned}$$

Practice

Simplify each of the following expressions, if possible.

1. $4^2 \times 4^4$
2. $x^4 \cdot x^{12}$
3. $(3x^2y^4)(9xy^5z)$
4. $(2xy)^2(4x^2y^3)$
5. $(3x)^5(2x)^2(3x^4)$
6. $x^3y^2z \cdot 4xy^2z^7$
7. $x^2y^3 + xy^2$
8. $(0.1xy)^4$
9. $(xyz)^6$
10. $2x^4(x^2 - y^2)$
11. $3x^5 - x^2$
12. $3x^8(x^2 - y^4)$

Expand and then simplify each of the following expressions.

13. $(x^5)^3$
14. $(x^6)^8$
15. $(x^a)^b$ *Hint: Look for a pattern in the previous two problems.*

6.2 Quotient Rules for Exponents

Here you'll learn how to divide two terms with the same base and find the power of a quotient.

Suppose you have the expression:

$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}$

How could you write this expression in a more concise way?

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and quotients.

RULE: To divide two powers with the same base, subtract the exponents.

$$\frac{a^m}{a^n} = \frac{\overbrace{(a \times a \times \ldots \times a)}^{m \text{ factors}}}{\underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}}} \quad m > n; a \neq 0$$

$$\frac{a^m}{a^n} = \underbrace{(a \times a \times \ldots \times a)}_{m-n \text{ factors}}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

RULE: To raise a quotient to a power, raise both the numerator and the denominator to the power.

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b}}$$

↓

n factors

n factors

↑

$$\left(\frac{a}{b}\right)^n = \frac{\overbrace{(a \times a \times \ldots \times a)}}{\underbrace{(b \times b \times \ldots \times b)}}$$

↓

n factors

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\ (b \neq 0)$$

Example A

Simplify $2^7 \div 2^3$.

Solution:

$2^7 \div 2^3$	The base is 2.
2^{7-3}	Keep the base of 2 and subtract the exponents.
2^4	The answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$2^4 = 2 \times 2 \times 2 \times 2$$
$$2^4 = 16$$
$$\boxed{2^7 \div 2^3 = 2^4 = 16}$$

Example B

Simplify $\frac{x^8}{x^2}$.

Solution:

$\frac{x^8}{x^2}$	The base is x.
x^{8-2}	Keep the base of x and subtract the exponents.
x^6	The answer is in exponential form.

$$\boxed{\frac{x^8}{x^2} = x^6}$$

Example C

Simplify $\frac{16x^5y^5}{4x^2y^3}$.

Solution:

$\frac{16x^5y^5}{4x^2y^3}$	The bases are x and y.
$4\left(\frac{x^5y^5}{x^2y^3}\right)$	Divide the coefficients - $16 \div 4 = 4$. Keep the base of x and y and subtract the exponents of the same base.
$4x^{5-2}y^{5-3}$	
$4x^{3}y^{2}$	

Concept Problem Revisited

$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}$ can be rewritten as $\frac{x^9y^5}{x^6y^3}$ and then simplified to x^3y^2.

Vocabulary

Base
In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent
In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.
In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents
The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

Simplify each of the following expressions.

1. $\left(\frac{2}{3}\right)^2$
2. $\left(\frac{x}{6}\right)^3$
3. $\left(\frac{3x}{4y}\right)^2$

Answers:

1. $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$
2. $\left(\frac{x}{6}\right)^3 = \frac{x^3}{6^3} = \frac{x^3}{216}$
3. $\left(\frac{3x}{4y}\right)^2 = \frac{3^2x^2}{4^2y^2} = \frac{9x^2}{16y^2}$

Practice

Simplify each of the following expressions, if possible.

1. $\left(\frac{2}{5}\right)^6$
2. $\left(\frac{4}{7}\right)^3$
3. $\left(\frac{x}{y}\right)^4$
4. $\frac{20x^4y^5}{5x^2y^4}$
5. $\frac{42x^2y^8z^2}{6xy^4z}$
6. $\left(\frac{3x}{4y}\right)^3$
7. $\frac{72x^2y^4}{8x^2y^3}$
8. $\left(\frac{x}{4}\right)^5$
9. $\frac{24x^{14}y^8}{3x^5y^7}$
10. $\frac{72x^3y^9}{24xy^6}$
11. $\left(\frac{7}{y}\right)^3$
12. $\frac{20x^{12}}{-5x^8}$

13. Simplify using the laws of exponents: $\frac{2^3}{2^5}$
14. Evaluate the numerator and denominator separately and then simplify the fraction: $\frac{2^3}{2^5}$
15. Use your result from the previous problem to determine the value of a: $\frac{2^3}{2^5} = \frac{1}{2^a}$
16. Use your results from the previous three problems to help you evaluate 2^{-4}.

6.3 Power Rule for Exponents

Here you'll learn how to find the power of a power.

Can you simplify an expression where an exponent has an exponent? For example, how would you simplify $[(2^3)^2]^4$?

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn a rule that has to do with raising a power to another power.

RULE: To raise a power to a new power, multiply the exponents.

$$(a^m)^n = \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}{}^n$$

$$(a^m)^n = \underbrace{\underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \times \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}}_{n \text{ times}}$$

$$(a^m)^n = \underbrace{a \times a \times a \ldots \times a}_{mn \text{ factors}}$$

$$(a^m)^n = a^{mn}$$

Example A

Evaluate $(2^3)^2$.

Solution: $(2^3)^2 = 2^6 = 64$.

Example B

Simplify $(x^7)^4$.

Solution: $(x^7)^4 = x^{28}$.

Example C

Evaluate $(3^2)^3$.

Solution: $(3^2)^3 = 3^6 = 729$.

Example D

Simplify $(x^2y^4)^2 \cdot (xy^4)^3$.

Solution: $(x^2y^4)^2 \cdot (xy^4)^3 = x^4y^8 \cdot x^3y^{12} = x^7y^{20}$.

Concept Problem Revisited

$[(2^3)^2]^4 = [2^6]^4 = 2^{24}$. Notice that the power rule applies even when a number has been raised to more than one power. The overall exponent is 24 which is $3 \cdot 2 \cdot 4$.

Vocabulary

Base
In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent
In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.
In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents
The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

You know you can rewrite 2^4 as $2 \times 2 \times 2 \times 2$ and then calculate in order to find that

$$\boxed{2^4 = 16}$$

. This concept can also be reversed. To write 32 as a power of 2, $32 = 2 \times 2 \times 2 \times 2 \times 2$. There are 5 twos; therefore,

$$\boxed{32 = 2^5}$$

. Use this idea to complete the following problems.

1. Write 81 as a power of 3.

2. Write $(9)^3$ as a power of 3.

3. Write $(4^3)^2$ as a power of 2.

Answers:

1. $81 = 3 \times 3 = 9 \times 3 = 27 \times 3 = 81$

There are 4 threes. Therefore

$$\boxed{81 = 3^4}$$

2. $9 = 3 \times 3 = 9$

There are 2 threes. Therefore

$$\boxed{9 = 3^2}$$

.
$(3^2)^3$ Apply the law of exponents for power to a power-multiply the exponents.
$3^{2\times 3} = 3^6$
Therefore

$$\boxed{(9)^3 = 3^6}$$

3. $4 = 2 \times 2 = 4$

There are 2 twos. Therefore

$$\boxed{4 = 2^2}$$

$\left((2^2)^3\right)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{2\times 3} = 2^6}$$

$(2^6)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{6\times 2} = 2^{12}}$$

Therefore

$$\boxed{(4^3)^2 = 2^{12}}$$

Practice

Simplify each of the following expressions.

1. $\left(\frac{x^4}{y^3}\right)^5$
2. $\frac{(5x^2y^4)^5}{(5xy^2)^3}$
3. $\frac{x^8y^9}{(x^2y)^3}$
4. $(x^2y^4)^3$
5. $(3x^2)^2 \cdot (4xy^4)^2$
6. $(2x^3y^5)(5x^2y)^3$

7. $(x^4y^6z^2)^2(3xyz)^3$
8. $\left(\frac{x^2}{2y^3}\right)^4$
9. $\frac{(4xy^3)^4}{(2xy^2)^3}$
10. True or false: $(x^2+y^3)^2 = x^4+y^6$
11. True or false: $(x^2y^3)^2 = x^4y^6$
12. Write 64 as a power of 4.
13. Write $(16)^3$ as a power of 2.
14. Write $(9^4)^2$ as a power of 3.
15. Write $(81)^2$ as a power of 3.
16. Write $(25^3)^4$ as a power of 5.

6.4 Exponential Growth Function

Here you'll learn how to analyze an exponential growth function and its graph.

A population of 10 mice grows at a rate of 300% every month. How many mice are in the population after six months?

Guidance

An **exponential function** has the variable in the exponent of the expression. All exponential functions have the form: $f(x) = a \cdot b^{x-h} + k$, where h and k move the function in the x and y directions respectively, much like the other functions we have seen in this text. b is the base and a changes how quickly or slowly the function grows. Let's take a look at the parent graph, $y = 2^x$.

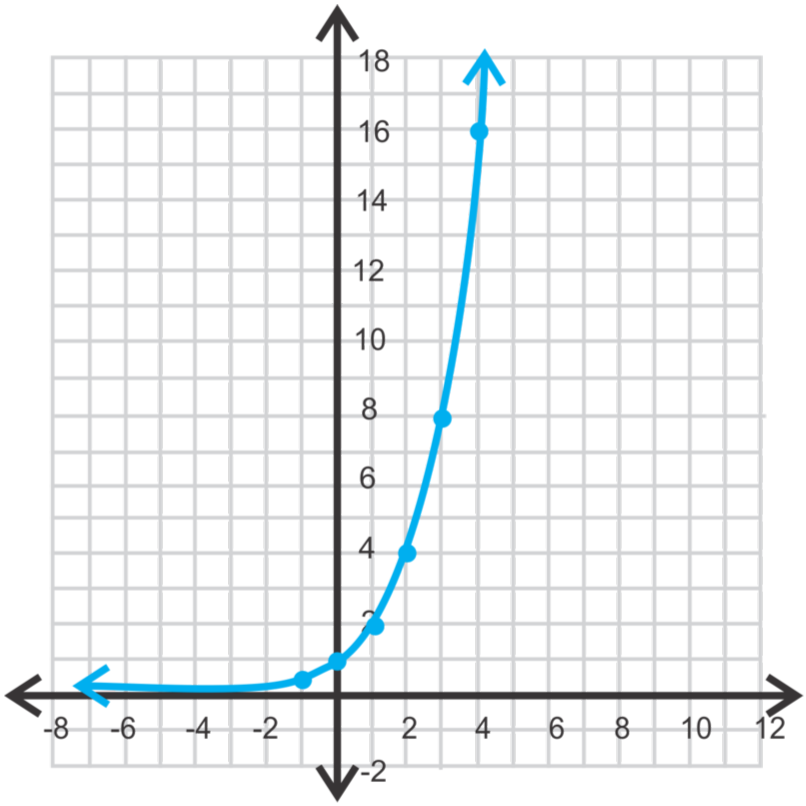
Example A

Graph $y = 2^x$. Find the y-intercept.

Solution: Let's start by making a table. Include some positive and negative values for x and zero.

TABLE 6.1:

x	2^x	y
3	2^3	8
2	2^2	4
1	2^1	2
0	2^0	1
-1	2^{-1}	$\frac{1}{2}$
-2	2^{-2}	$\frac{1}{4}$
-3	2^{-3}	$\frac{1}{8}$



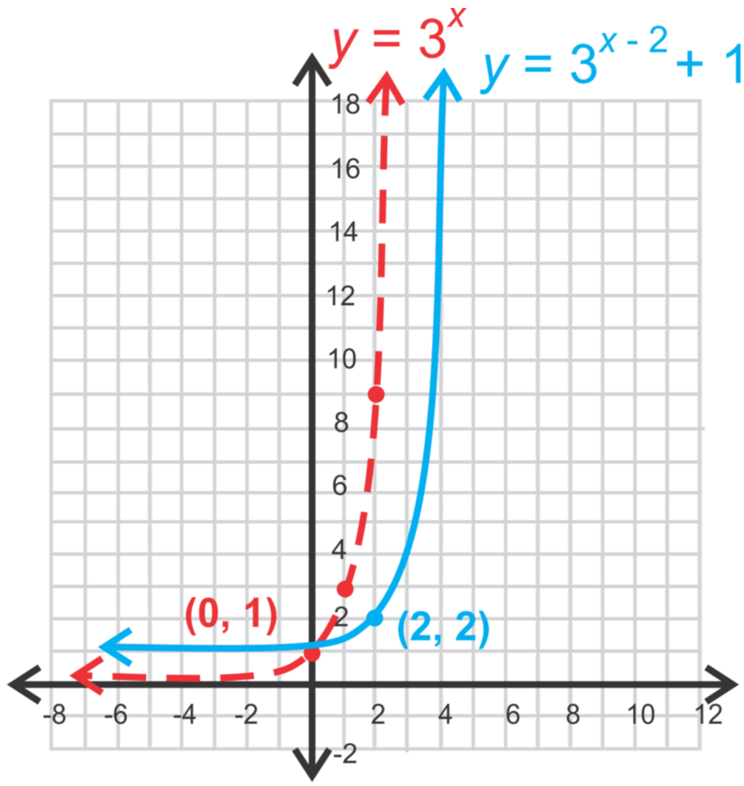
This is the typical shape of an **exponential growth function**. The function grows "exponentially fast". Meaning, in this case, the function grows in powers of 2. For an exponential function to be a growth function, $a > 0$ and $b > 1$ and h and k are both zero $(y = ab^x)$. From the table, we see that the y-intercept is (0, 1).

Notice that the function gets very, very close to the x-axis, but never touches or passes through it. Even if we chose $x = -50,\ y$ would be $2^{-50} = \frac{1}{2^{50}}$, which is still not zero, but very close. In fact, the function will never reach zero, even though it will get smaller and smaller. Therefore, this function approaches the line $y = 0$, but will never touch or pass through it. This type of boundary line is called an **asymptote**. In the case with all exponential functions, there will be a horizontal asymptote. If $k = 0$, then the asymptote will be $y = 0$.

Example B

Graph $y = 3^{x-2} + 1$. Find the y-intercept, asymptote, domain and range.

Solution: This is not considered a growth function because h and k are not zero. To graph something like this (without a calculator), start by graphing $y = 3^x$ and then shift it h units in the x-direction and k units in the y-direction.



Notice that the point (0, 1) from $y = 3^x$ gets shifted to the right 2 units and up one unit and is (2, 2) in the translated function, $y = 3^{x-2} + 1$. Therefore, the asymptote is $y = 1$. To find the y-intercept, plug in $x = 0$.

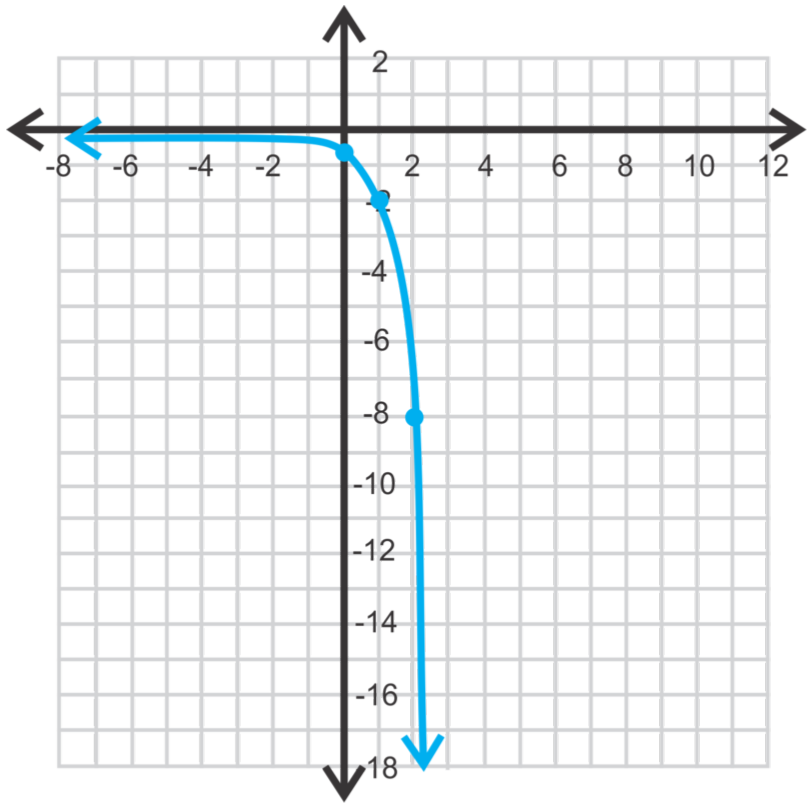
$$y = 3^{0-2} + 1 = 3^{-2} + 1 = 1\frac{1}{9} = 1.\overline{1}$$

The domain of all exponential functions is all real numbers. The range will be everything greater than the asymptote. In this example, the range is $y > 1$.

Example C

Graph the function $y = -\frac{1}{2} \cdot 4^x$. Determine if it is an exponential growth function.

Solution: In this example, we will outline how to use the graphing calculator to graph an exponential function. First, clear out anything in Y=. Next, input the function into Y1= -(1/2)4^X and press **GRAPH**. Adjust your window accordingly.



This is not an exponential growth function, because it does not grow in a positive direction. By looking at the definition of a growth function, $a > 0$, and it is not here.

Intro Problem Revisit

This is an example of exponential growth, so we can use the exponential form $f(x) = a \cdot b^{x-h} + k$. In this case, a = 10, the starting population; b = 300% or 3, the rate of growth; x-h = 6 the number of months, and k = 0.

$$\begin{aligned} P &= 10 \cdot 3^6 \\ &= 10 \cdot 729 = 7290 \end{aligned}$$

Therefore, the mouse population after six months is 7,290.

Guided Practice

Graph the following exponential functions. Determine if they are growth functions. Then, find the y-intercept, asymptote, domain and range. Use an appropriate window.

1. $y = 3^{x-4} - 2$

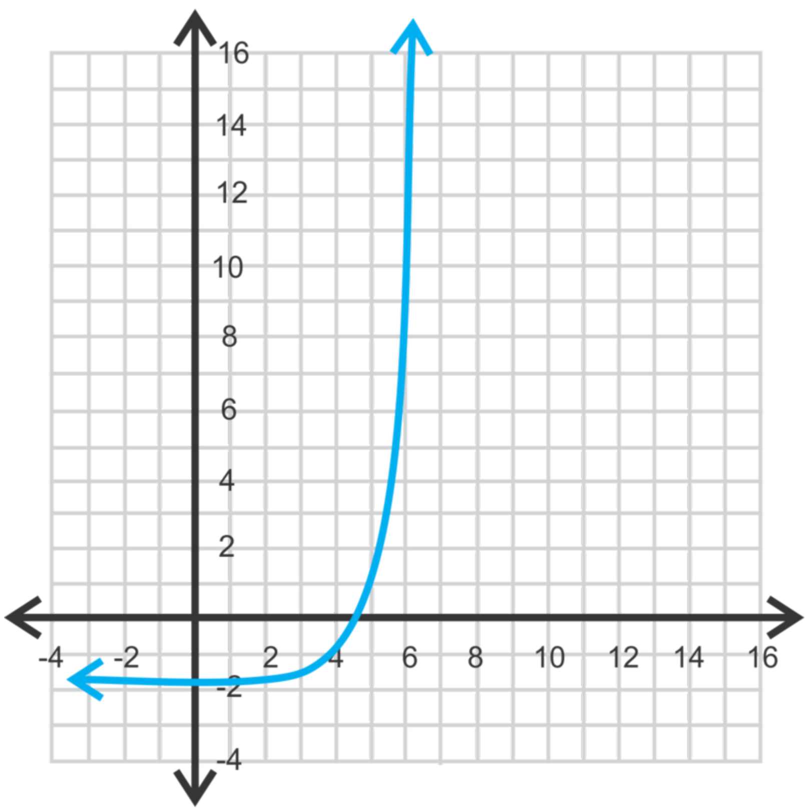
2. $f(x) = (-2)^{x+5}$

3. $f(x) = 5^x$

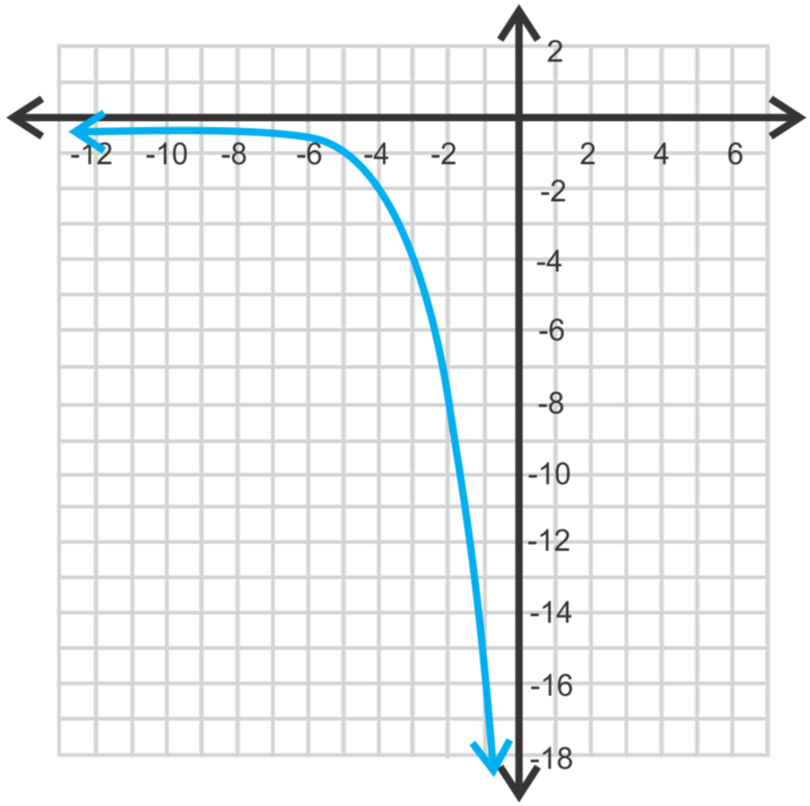
4. Abigail is in a singles tennis tournament. She finds out that there are eight rounds until the final match. If the tournament is single elimination, how many games will be played? How many competitors are in the tournament?

Answers

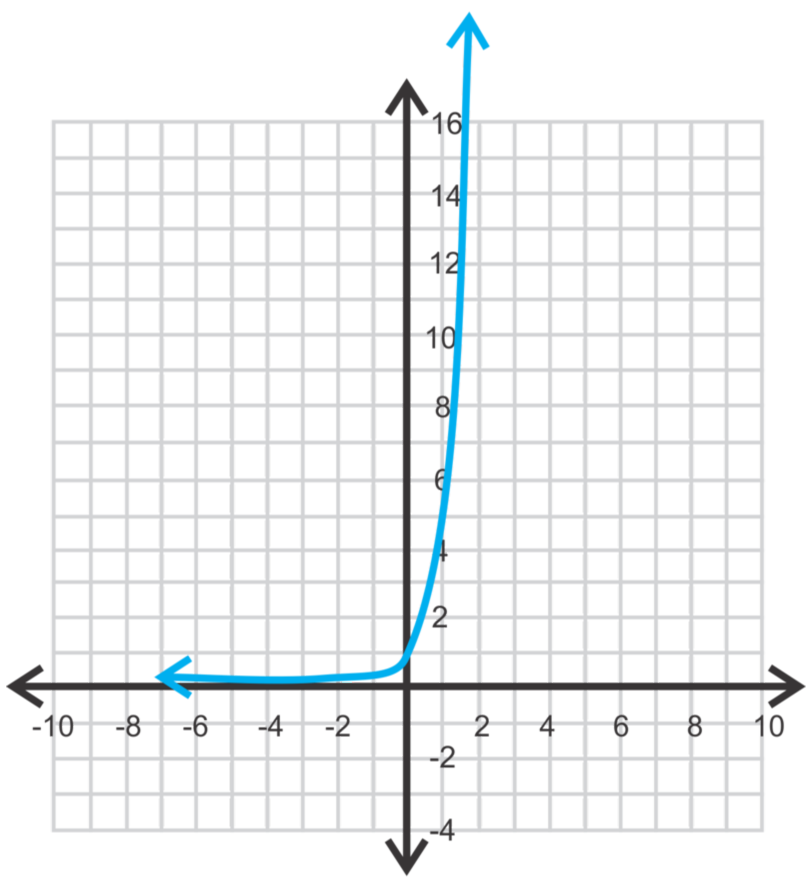
1. This is not a growth function because h and k are not zero. The y-intercept is $y = 3^{0-4} - 2 = \frac{1}{81} - 2 = -1\frac{80}{81}$, the asymptote is at $y = -2$, the domain is all real numbers and the range is $y > -2$.



2. This is not a growth function because h is not zero. The y-intercept is $y = (-2)^{0+5} = (-2)^5 = -32$, the asymptote is at $y = 0$, the domain is all real numbers and the range is $y > 0$.



3. This is a growth function. The y-intercept is $y = 5^{\circ} = 1$, the asymptote is at $y = 0$, the domain is all real numbers and the range is $y > 0$.



4. If there are eight rounds to single's games, there are will be $2^8 = 256$ competitors. In the first round, there will be 128 matches, then 64 matches, followed by 32 matches, then 16 matches, 8, 4, 2, and finally the championship game. Adding all these all together, there will be $128+64+32+16+8+4+2+1$ or 255 total matches.

Vocabulary

Exponential Function
A function whose variable is in the exponent. The general form is $y = a \cdot b^{x-h} + k$.

Exponential Growth Function
A specific type of exponential function where $h = k = 0, a > 0$, and $b > 1$. The general form is $y = ab^x$.

Asymptote
A boundary line that restricts the domain or range. This line is not part of the graph.

Practice

Graph the following exponential functions. Find the y-intercept, the equation of the asymptote and the domain and range for each function.

1. $y = 4^x$
2. $y = (-1)(5)^x$
3. $y = 3^x - 2$
4. $y = 2^x + 1$
5. $y = 6^{x+3}$
6. $y = -\frac{1}{4}(2)^x + 3$
7. $y = 7^{x+3} - 5$
8. $y = -(3)^{x-4} + 2$
9. $y = 3(2)^{x+1} - 5$
10. What is the y-intercept of $y = a^x$? Why is that?
11. What is the range of the function $y = a^{x-h} + k$?
12. March Madness is a single-game elimination tournament of 64 college basketball teams. How many games will be played until there is a champion? Include the championship game.
13. In 2012, the tournament added 4 teams to make it a field of 68 and there are 4 "play-in" games at the beginning of the tournament. How many games are played now?
14. An investment grows according the function $A = P(1.05)^t$ where P represents the initial investment, A represents the value of the investment and t represents the number of years of investment. If \$10,000 was the initial investment, how much would the value of the investment be after 10 years, to the nearest dollar?
15. How much would the value of the investment be after 20 years, to the nearest dollar?