CHAPTER 7

Chapter 7: Polynomials

Chapter Outline

7.1 Polynomials in Standard Form

Here you'll learn how to identify polynomials and find their degree. You'll also learn how to write polynomial expressions in standard form and simplify them by combining like terms.

What if you were given an algebraic expression like $3x-2x^2+5-x+6x^2$? How could you simplify it and find its degree? After completing this Concept, you'll be able to combine like terms to simplify polynomial expressions like this one and classify them by degree.

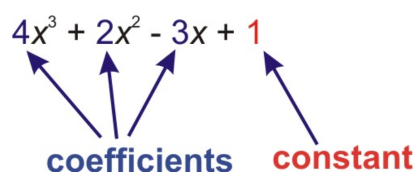
Guidance

So far we've seen functions described by straight lines (linear functions) and functions where the variable appeared in the exponent (exponential functions). In this section we'll introduce polynomial functions. A **polynomial** is made up of different terms that contain **positive integer** powers of the variables. Here is an example of a polynomial:

$$4x^3+2x^2-3x+1$$

Each part of the polynomial that is added or subtracted is called a **term** of the polynomial. The example above is a polynomial with *four terms*.

The numbers appearing in each term in front of the variable are called the **coefficients**. The number appearing all by itself without a variable is called a **constant**.



In this case the coefficient of x^3 is **4**, the coefficient of x^2 is **2**, the coefficient of x is **-3** and the constant is **1**.

Degrees of Polynomials and Standard Form

Each term in the polynomial has a different **degree**. The degree of the term is the power of the variable in that term.

$4x^3$	has degree 3 and is called a cubic term or 3^{rd} order term.
$2x^2$	has degree 2 and is called a quadratic term or 2^{nd} order term.
$-3x$	has degree 1 and is called a linear term or 1^{st} order term.
1	has degree 0 and is called the constant.

By definition, **the degree of the polynomial** is the same as the degree of the term with the highest degree. This example is a polynomial of degree 3, which is also called a "cubic" polynomial. (Why do you think it is called a cubic?).

Polynomials can have more than one variable. Here is another example of a polynomial:

$$t^4-6s^3t^2-12st+4s^4-5$$

This is a polynomial because all the exponents on the variables are positive integers. This polynomial has five terms. Let's look at each term more closely.

Note: *The degree of a term is the sum of the powers on each variable in the term.* In other words, the degree of each term is the number of variables that are multiplied together in that term, whether those variables are the same or different.

t^4	has a degree of 4, so it's a 4^{th} order term
$-6s^3t^2$	has a degree of 5, so it's a 5^{th} order term.
$-12st$	has a degree of 2, so it's a 2^{nd} order term.
$4s^4$	has a degree of 4, so it's a 4^{th} order term.
-5	is a constant, so its degree is 0.

Since the highest degree of a term in this polynomial is 5, then this is polynomial of degree 5^{th} or a 5^{th} order polynomial.

A polynomial that has only one term has a special name. It is called a **monomial** (*mono* means one). A monomial can be a constant, a variable, or a product of a constant and one or more variables. You can see that each term in a polynomial is a monomial, so a polynomial is just the sum of several monomials. Here are some examples of monomials:

$$b^2 \qquad -2ab^2 \qquad 8 \qquad \frac{1}{4}x^4 \qquad -29xy$$

Example A

For the following polynomials, identify the coefficient of each term, the constant, the degree of each term and the degree of the polynomial.

a) $x^5-3x^3+4x^2-5x+7$

b) $x^4-3x^3y^2+8x-12$

Solution

a) $x^5-3x^3+4x^2-5x+7$

The coefficients of each term in order are 1, -3, 4, and -5 and the constant is 7.

The degrees of each term are 5, 3, 2, 1, and 0. Therefore the degree of the polynomial is 5.

b) $x^4-3x^3y^2+8x-12$

The coefficients of each term in order are 1, -3, and 8 and the constant is -12.

The degrees of each term are 4, 5, 1, and 0. Therefore the degree of the polynomial is 5.

Example B

Identify the following expressions as polynomials or non-polynomials.

a) $5x^5-2x$

b) $3x^2-2x^{-2}$

c) $x\sqrt{x}-1$

d) $\frac{5}{x^3+1}$

e) $4x^{\frac{1}{3}}$

f) $4xy^2 - 2x^2y - 3 + y^3 - 3x^3$

Solution

a) This ***is*** a polynomial.

b) This is ***not*** a polynomial because it has a negative exponent.

c) This is ***not*** a polynomial because it has a radical.

d) This is ***not*** a polynomial because the power of x appears in the denominator of a fraction (and there is no way to rewrite it so that it does not).

e) This is ***not*** a polynomial because it has a fractional exponent.

f) This ***is*** a polynomial.

Often, we arrange the terms in a polynomial in order of decreasing power. This is called **standard form**.

The following polynomials are in standard form:

$$4x^4 - 3x^3 + 2x^2 - x + 1$$

$$a^4b^3 - 2a^3b^3 + 3a^4b - 5ab^2 + 2$$

The first term of a polynomial in standard form is called the **leading term**, and the coefficient of the leading term is called the **leading coefficient**.

The first polynomial above has the leading term $4x^4$, and the leading coefficient is 4.

The second polynomial above has the leading term a^4b^3, and the leading coefficient is 1.

Example C

Rearrange the terms in the following polynomials so that they are in standard form. Indicate the leading term and leading coefficient of each polynomial.

a) $7 - 3x^3 + 4x$

b) $ab - a^3 + 2b$

c) $-4b + 4 + b^2$

Solution

a) $7 - 3x^3 + 4x$ becomes $-3x^3 + 4x + 7$. Leading term is $-3x^3$; leading coefficient is -3.

b) $ab - a^3 + 2b$ becomes $-a^3 + ab + 2b$. Leading term is $-a^3$; leading coefficient is -1.

c) $-4b + 4 + b^2$ becomes $b^2 - 4b + 4$. Leading term is b^2; leading coefficient is 1.

Simplifying Polynomials

A polynomial is simplified if it has no terms that are alike. **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, whether they have the same or different coefficients.

For example, $2x^2y$ and $5x^2y$ are like terms, but $6x^2y$ and $6xy^2$ are not like terms.

When a polynomial has like terms, we can simplify it by combining those terms.

$$x^2 + \underline{6xy} - \underline{4xy} + y^2$$

$\nearrow \quad \nwarrow$

Like terms

We can simplify this polynomial by combining the like terms $6xy$ and $-4xy$ into $(6-4)xy$, or $2xy$. The new polynomial is $x^2 + 2xy + y^2$.

Example D

Simplify the following polynomials by collecting like terms and combining them.

a) $2x - 4x^2 + 6 + x^2 - 4 + 4x$

b) $a^3b^3 - 5ab^4 + 2a^3b - a^3b^3 + 3ab^4 - a^2b$

Solution

a) Rearrange the terms so that like terms are grouped together: $(-4x^2 + x^2) + (2x + 4x) + (6 - 4)$

Combine each set of like terms: $-3x^2 + 6x + 2$

b) Rearrange the terms so that like terms are grouped together: $(a^3b^3 - a^3b^3) + (-5ab^4 + 3ab^4) + 2a^3b - a^2b$

Combine each set of like terms: $0 - 2ab^4 + 2a^3b - a^2b = -2ab^4 + 2a^3b - a^2b$

Vocabulary

- A ***polynomial*** is an expression made with constants, variables, and *positive integer* exponents of the variables.
- In a polynomial, the number appearing in each term in front of the variables is called the **coefficient.**
- In a polynomial, the number appearing all by itself without a variable is called the **constant.**
- A **monomial** is a one-termed polynomial. It can be a constant, a variable, or a variable with a coefficient.
- The **degree of a polynomial** is the largest degree of the terms. The **degree of a term** is the power of the variable, or if the term has more than one variable, it is the sum of the powers on each variable.
- We arrange the terms in a polynomial in **standard form** in which the term with the highest degree is first and is followed by the other terms in order of decreasing powers.
- **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, but they can have different coefficients.

Guided Practice

Simplify and rewrite the following polynomial in standard form. State the degree of the polynomial.

$16x^2y^3 - 3xy^5 - 2x^3y^2 + 2xy - 7x^2y^3 + 2x^3y^2$

Solution:

Start by simplifying by combining like terms:

$16x^2y^3 - 3xy^5 - 2x^3y^2 + 2xy - 7x^2y^3 + 2x^3y^2$

is equal to

$(16x^2y^3 - 7x^2y^3) - 3xy^5 + (-2x^3y^2 + 2x^3y^2) + 2xy$

which simplifies to

$9x^2y^3 - 3xy^5 + 2xy$.

In order to rewrite in standard form, we need to determine the degree of each term. The first term has a degree of $2+3=5$, the second term has a degree of $1+5=6$, and the last term has a degree of $1+1=2$. We will rewrite the terms in order from largest degree to smallest degree:

$-3xy^5 + 9x^2y^3 + 2xy$

The degree of a polynomial is the largest degree of all the terms. In this case that is 6.

Practice

Indicate whether each expression is a polynomial.

1. $x^2 + 3x^{\frac{1}{2}}$
2. $\frac{1}{3}x^2y - 9y^2$
3. $3x^{-3}$
4. $\frac{2}{3}t^2 - \frac{1}{t^2}$
5. $\sqrt{x} - 2x$
6. $\left(x^{\frac{3}{2}}\right)^2$

Express each polynomial in standard form. Give the degree of each polynomial.

7. $3 - 2x$
8. $8 - 4x + 3x^3$
9. $-5 + 2x - 5x^2 + 8x^3$
10. $x^2 - 9x^4 + 12$
11. $5x + 2x^2 - 3x$

7.2 Adding and Subtracting Polynomials

Here you'll learn how to add and subtract polynomials, as well as learn about the different parts of a polynomial.

Rectangular prism A has a volume of x^3+2x^2-3. Rectangular prism B has a volume of $x^4+2x^3-8x^2$. How much larger is the volume of rectangular prism B than rectangular prism A?

Guidance

A **polynomial** is an expression with multiple variable terms, such that the exponents are greater than or equal to zero. All quadratic and linear equations are polynomials. Equations with negative exponents, square roots, or variables in the denominator are not polynomials.

Polynomials	Not Polynomials
$2x^2+6x-9$	$10x^{-1}+6x^2$
$-x^3+9$	$\sqrt{x}-2$
$4x^4+5x^3-8x^2+12x+24$	$\frac{3}{x}+5$

Now that we have established what a polynomial is, there are a few important parts. Just like with a quadratic, a polynomial can have a **constant**, which is a number without a variable. The **degree** of a polynomial is the largest exponent. For example, all quadratic equations have a degree of 2. Lastly, the **leading coefficient** is the coefficient in front of the variable with the degree. In the polynomial $4x^4+5x^3-8x^2+12x+24$ above, the degree is 4 and the leading coefficient is also 4. Make sure that when finding the degree and leading coefficient you have the polynomial in standard form. **Standard form** lists all the variables in order, from greatest to least.

Example A

Rewrite $x^3-5x^2+12x^4+15-8x$ in standard form and find the degree and leading coefficient.

Solution: To rewrite in standard form, put each term in order, from greatest to least, according to the exponent. Always write the constant last.

$$x^3-5x^2+12x^4+15-8x \rightarrow 12x^4+x^3-5x^2-8x+15$$

Now, it is easy to see the leading coefficient, 12, and the degree, 4.

Example B

Simplify $(4x^3-2x^2+4x+15)+(x^4-8x^3-9x-6)$

Solution: To add or subtract two polynomials, combine like terms. **Like terms** are any terms where the exponents of the variable are the same. We will regroup the polynomial to show the like terms.

$$(4x^3 - 2x^2 + 4x + 15) + (x^4 - 8x^3 - 9x - 6)$$
$$x^4 + (4x^3 - 8x^3) - 2x^2 + (4x - 9x) + (15 - 6)$$
$$x^4 - 4x^3 - 2x^2 - 5x + 9$$

Example C

Simplify $(2x^3 + x^2 - 6x - 7) - (5x^3 - 3x^2 + 10x - 12)$

Solution: When subtracting, distribute the negative sign to every term in the second polynomial, then combine like terms.

$$(2x^3 + x^2 - 6x - 7) - (5x^3 - 3x^2 + 10x - 12)$$
$$2x^3 + x^2 - 6x - 7 - 5x^3 + 3x^2 - 10x + 12$$
$$(2x^3 - 5x^3) + (x^2 + 3x^2) + (-6x - 10x) + (-7 + 12)$$
$$-3x^3 + 4x^2 - 16x + 5$$

Intro Problem Revisit

We need to subtract the volume of rectangular prism A from the volume of rectangular prism B.

$$\begin{aligned}(x^4 + 2x^3 - 8x^2) - (x^3 + 2x^2 - 3) \\ = x^4 + 2x^3 - 8x^2 - x^3 - 2x^2 + 3 \\ = x^4 + x^3 - 10x^2 + 3\end{aligned}$$

.

Therefore, the difference between the two volumes is $x^4 + x^3 - 10x^2 + 3$.

Guided Practice

1. Is $\sqrt{2x^3 - 5x} + 6$ a polynomial? Why or why not?

2. Find the leading coefficient and degree of $6x^2 - 3x^5 + 16x^4 + 10x - 24$.

Add or subtract.

3. $(9x^2 + 4x^3 - 15x + 22) + (6x^3 - 4x^2 + 8x - 14)$

4. $(7x^3 + 20x - 3) - (x^3 - 2x^2 + 14x - 18)$

Answers

1. No, this is not a polynomial because x is under a square root in the equation.

2. In standard form, this polynomial is $-3x^5 + 16x^4 + 6x^2 + 10x - 24$. Therefore, the degree is 5 and the leading coefficient is -3.

3. $(9x^2 + 4x^3 - 15x + 22) + (6x^3 - 4x^2 + 8x - 14) = 10x^3 + 5x^2 - 7x + 8$

4. $(7x^3 + 20x - 3) - (x^3 - 2x^2 + 14x - 18) = 6x^3 + 2x^2 + 6x + 15$

Vocabulary

Polynomial
An expression with multiple variable terms, such that the exponents are greater than or equal to zero.

Constant
A number without a variable in a mathematical expression.

Degree(of a polynomial)
The largest exponent in a polynomial.

Leading coefficient
The coefficient in front of the variable with the degree.

Standard form
Lists all the variables in order, from greatest to least.

Like terms
Any terms where the exponents of the variable are the same.

Practice

Determine if the following expressions are polynomials. If not, state why. If so, write in standard form and find the degree and leading coefficient.

1. $\frac{1}{x^2} + x + 5$
2. $x^3 + 8x^4 - 15x + 14x^2 - 20$
3. $x^3 + 8$
4. $5x^{-2} + 9x^{-1} + 16$
5. $x^2\sqrt{2} - x\sqrt{6} + 10$
6. $\frac{x^4 + 8x^2 + 12}{3}$
7. $\frac{x^2 - 4}{x}$
8. $-6x^3 + 7x^5 - 10x^6 + 19x^2 - 3x + 41$

Add or subtract the following polynomials.

9. $(x^3 + 8x^2 - 15x + 11) + (3x^3 - 5x^2 - 4x + 9)$
10. $(-2x^4 + x^3 + 12x^2 + 6x - 18) - (4x^4 - 7x^3 + 14x^2 + 18x - 25)$
11. $(10x^3 - x^2 + 6x + 3) + (x^4 - 3x^3 + 8x^2 - 9x + 16)$
12. $(7x^3 - 2x^2 + 4x - 5) - (6x^4 + 10x^3 + x^2 + 4x - 1)$
13. $(15x^2 + x - 27) + (3x^3 - 12x + 16)$
14. $(2x^5 - 3x^4 + 21x^2 + 11x - 32) - (x^4 - 3x^3 - 9x^2 + 14x - 15)$
15. $(8x^3 - 13x^2 + 24) - (x^3 + 4x^2 - 2x + 17) + (5x^2 + 18x - 19)$

7.3 Multiplication of Polynomials

Here you will learn how to multiply polynomials using the distributive property.

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

Guidance

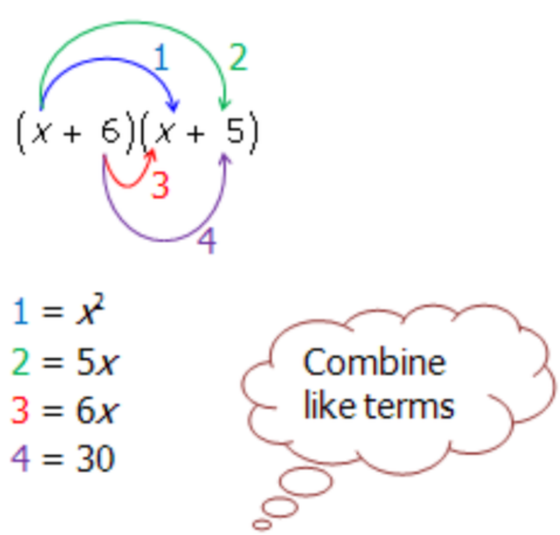
To multiply polynomials you will need to use the distributive property. Recall that the distributive property says that if you start with an expression like $3(5x+2)$, you can simplify it by multiplying both terms inside the parentheses by 3 to get a final answer of $15x+6$.

When multiplying polynomials, you will need to use the distributive property more than once for each problem.

Example A

Find the product: $(x+6)(x+5)$

Solution: To answer this question you will use the distributive property. The distributive property would tell you to multiply x in the first set of parentheses by everything inside the second set of parentheses , then multiply 6 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



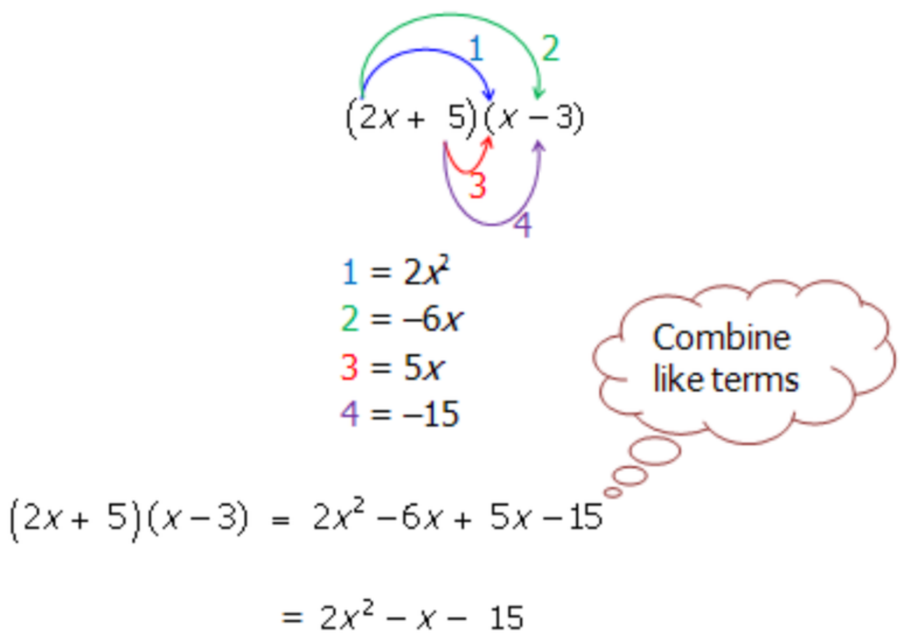
$$(x+6)(x+5) = x^2 + 5x + 6x + 30$$

$$= x^2 + 11x + 30$$

Example B

Find the product: $(2x+5)(x-3)$

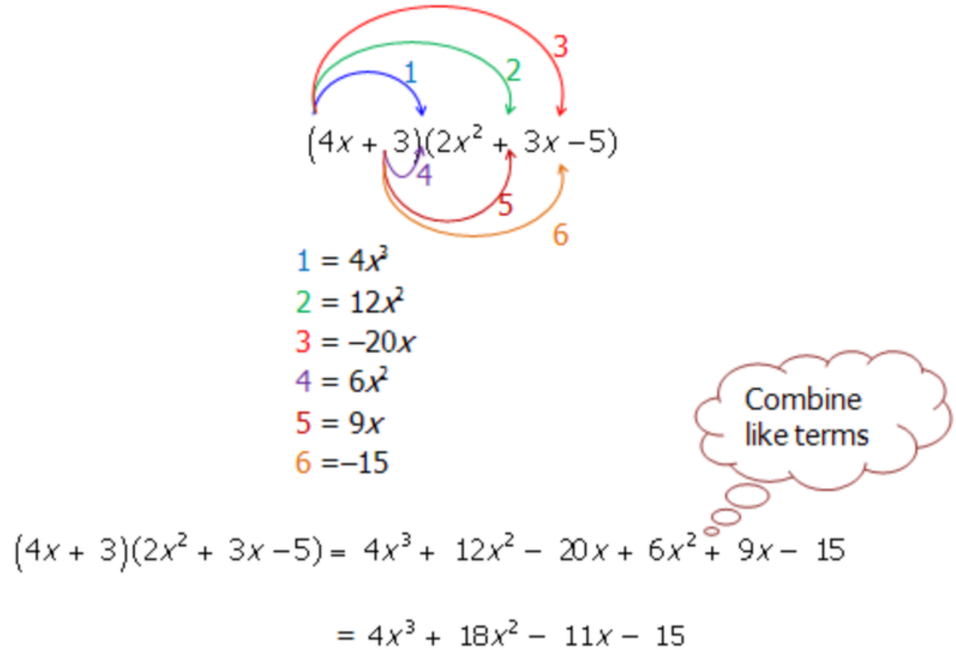
Solution: Again, use the distributive property. The distributive property tells you to multiply $2x$ in the first set of parentheses by everything inside the second set of parentheses , then multiply 5 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



Example C

Find the product: $(4x+3)(2x^2+3x-5)$

Solution: Even though at first this question may seem different, you can still use the distributive property to find the product. The distributive property tells you to multiply $4x$ in the first set of parentheses by everything inside the second set of parentheses, then multiply 3 in the first set of parentheses by everything in the second set of parentheses. Here is what that looks like:



Concept Problem Revisited

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

What is known?

The height of the picture frame is $4x+7$The equations:

- The width of the picture frame is $3x+5$

The formula:

$$\begin{aligned}
\text{Area} &= w \times h \\
\text{Area} &= (3x+5)(4x+7) \\
\text{Area} &= 12x^2+21x+20x+35 \\
\text{Area} &= 12x^2+41x+35
\end{aligned}$$

Vocabulary

Distributive Property

The ***distributive property*** states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, in the expression: $\frac{2}{3}(x+5)$, the distributive property states that the product of a number $\left(\frac{2}{3}\right)$ and a sum $(x+5)$ is equal to the sum of the individual products of the number $\left(\frac{2}{3}\right)$ and the addends $(x$ and $5)$.

Like Terms

Like terms refers to terms in which the degrees match and the variables match. For example $3x$ and $4x$ are like terms. Like terms are also known as **similar terms**.

Guided Practice

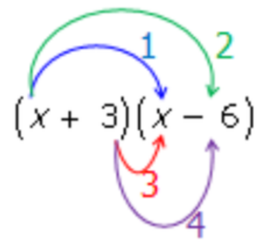
1. Find the product: $(x+3)(x-6)$

2. Find the product: $(2x+5)(3x^2-2x-7)$

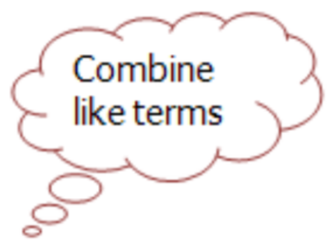
3. An average football field has the dimensions of 160 ft by 360 ft. If the expressions to find these dimensions were $(3x+7)$ and $(7x+3)$, what value of x would give the dimensions of the football field?

Answers:

1. $(x+3)(x-6)$

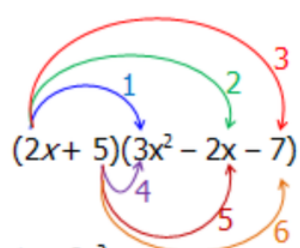


$1 = x^2$
$2 = -6x$
$3 = 3x$
$4 = -18$



$$(x+3)(x-6) = x^2 - 6x + 3x - 18$$
$$= x^2 - 3x - 18$$

2. $(2x+5)(3x^2-2x-7)$



$1 = 6x^3$
$2 = -4x^2$
$3 = -14x$
$4 = 15x^2$
$5 = -10x$
$6 = -35$

$$(4x+3)(2x^2+3x-5) = 6x^3 - 4x^2 - 14x + 15x^2 - 10x - 35$$
$$= 6x^3 + 11x^2 - 24x - 35$$

3. Area $= l \times w$

$$\begin{aligned}
\text{Area} &= 360 \times 160 \\
(7x+3) &= 360 \\
7x &= 360-3 \\
7x &= 357 \\
x &= 51 \\
\\
(3x+7) &= 160 \\
3x &= 160-7 \\
3x &= 153 \\
x &= 51
\end{aligned}$$

The value of x that satisfies these expressions is 51.

Practice

Use the distributive property to find the product of each of the following polynomials:

1. $(x+4)(x+6)$
2. $(x+3)(x+5)$
3. $(x+7)(x-8)$
4. $(x-9)(x-5)$
5. $(x-4)(x-7)$
6. $(x+3)(x^2+x+5)$
7. $(x+7)(x^2-3x+6)$
8. $(2x+5)(x^2-8x+3)$
9. $(2x-3)(3x^2+7x+6)$
10. $(5x-4)(4x^2-8x+5)$
11. $9a^2(6a^3+3a+7)$
12. $-4s^2(3s^3+7s^2+11)$
13. $(x+5)(5x^3+2x^2+3x+9)$
14. $(t-3)(6t^3+11t^2+22)$
15. $(2g-5)(3g^3+9g^2+7g+12)$

7.4 Special Products of Polynomials

Here you'll learn how to find two special polynomial products: 1) the square of a binomial and 2) two binomials where the sum and difference formula can be applied. You'll also learn how to apply special products of polynomials to solve real-world problems.

What if you wanted to multiply two binomials that were exactly the same, like $(x^2-2)(x^2-2)$? Similarly what if you wanted to multiply two binomials in which the sign between the two terms was the opposite in one from the other, like $(x^2-2)(x^2+2)$? What shortcuts could you use? After completing this Concept, you'll be able to find the square of a binomial as well as the product of binomials using the sum and difference formula.

Guidance

We saw that when we multiply two binomials we need to make sure to multiply each term in the first binomial with each term in the second binomial. Let's look at another example.

Multiply two linear binomials (binomials whose degree is 1):

$$(2x+3)(x+4)$$

When we multiply, we obtain a quadratic polynomial (one with degree 2) with four terms:

$$2x^2+8x+3x+12$$

The middle terms are like terms and we can combine them. We simplify and get $2x^2+11x+12$. This is a quadratic, or second-degree, **trinomial** (polynomial with three terms).

You can see that every time we multiply two linear binomials with one variable, we will obtain a quadratic polynomial. In this section we'll talk about some special products of binomials.

Find the Square of a Binomial

One special binomial product is the **square of a binomial**. Consider the product $(x+4)(x+4)$.

Since we are multiplying the same expression by itself, that means we are squaring the expression. $(x+4)(x+4)$ is the same as $(x+4)^2$.

When we multiply it out, we get $x^2+4x+4x+16$, which simplifies to $x^2+8x+16$.

Notice that the two middle terms—the ones we added together to get $8x$—were the same. Is this a coincidence? In order to find that out, let's square a general linear binomial.

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Sure enough, the middle terms are the same. How about if the expression we square is a difference instead of a sum?

$$\begin{aligned}(a-b)^2 = (a-b)(a-b) &= a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2\end{aligned}$$

It looks like the middle two terms are the same in general whenever we square a binomial. The general pattern is: to square a binomial, take the square of the first term, add or subtract twice the product of the terms, and add the square of the second term. You should remember these formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$
and
$$(a-b)^2 = a^2 - 2ab + b^2$$

Remember! Raising a polynomial to a power means that we multiply the polynomial by itself however many times the exponent indicates. For instance, $(a+b)^2 = (a+b)(a+b)$. **Don't make the common mistake of thinking that** $(a+b)^2 = a^2 + b^2$**!** To see why that's not true, try substituting numbers for a and b into the equation (for example, $a = 4$ and $b = 3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

We can apply the formulas for squaring binomials to any number of problems.

Example A

Square each binomial and simplify.

a) $(x+10)^2$

b) $(2x-3)^2$

c) $(x^2+4)^2$

Solution

Let's use the square of a binomial formula to multiply each expression.

a) $(x+10)^2$

If we let $a = x$ and $b = 10$, then our formula $(a+b)^2 = a^2 + 2ab + b^2$ becomes $(x+10)^2 = x^2 + 2(x)(10) + 10^2$, which simplifies to $x^2 + 20x + 100$.

b) $(2x-3)^2$

If we let $a = 2x$ and $b = 3$, then our formula $(a-b)^2 = a^2 - 2ab + b^2$ becomes $(2x-3)^2 = (2x^2) - 2(2x)(3) + (3)^2$, which simplifies to $4x^2 - 12x + 9$.

c) $(x^2+4)^2$

If we let $a = x^2$ and $b = 4$, then

$$\begin{aligned}(x^2+4)^2 &= (x^2)^2 + 2(x^2)(4) + (4)^2 \\ &= x^4 + 8x^2 + 16\end{aligned}$$

Find the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$\begin{aligned}(x+4)(x-4) &= x^2-4x+4x-16\\ &= x^2-16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they *cancel out* when we collect like terms. This is not a coincidence. This always happens when we multiply a sum and difference of the same terms. In general,

$$\begin{aligned}(a+b)(a-b) &= a^2-ab+ab-b^2\\ &= a^2-b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula: $(a+b)(a-b)=a^2-b^2$

Let's apply this formula to a few examples.

Example B

Multiply the following binomials and simplify.

a) $(x+3)(x-3)$

b) $(5x+9)(5x-9)$

c) $(2x^3+7)(2x^3-7)$

Solution

a) Let $a=x$ and $b=3$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\ (x+3)(x-3) &= x^2-3^2\\ &= x^2-9\end{aligned}$$

b) Let $a=5x$ and $b=9$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\ (5x+9)(5x-9) &= (5x)^2-9^2\\ &= 25x^2-81\end{aligned}$$

c) Let $a=2x^3$ and $b=7$, then:

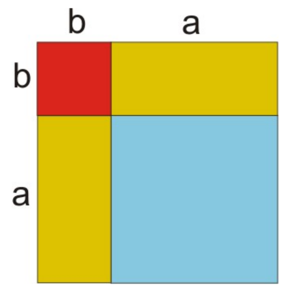
$$\begin{aligned}(2x^3+7)(2x^3-7) &= (2x^3)^2-(7)^2\\ &= 4x^6-49\end{aligned}$$

Solve Real-World Problems Using Special Products of Polynomials

Now let's see how special products of polynomials apply to geometry problems and to mental arithmetic.

Example C

Find the area of the following square:



Solution

The length of each side is $(a+b)$, so the area is $(a+b)(a+b)$.

Notice that this gives a visual explanation of the square of a binomial. The blue square has area a^2, the red square has area b^2, and each rectangle has area ab, so added all together, the area $(a+b)(a+b)$ is equal to $a^2+2ab+b^2$.

The next example shows how you can use the special products to do fast mental calculations.

Example D

Use the difference of squares and the binomial square formulas to find the products of the following numbers without using a calculator.

a) 43×57

b) 45^2

c) 481×319

Solution

The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

a) Rewrite 43 as $(50-7)$ and 57 as $(50+7)$.

Then $43 \times 57 = (50-7)(50+7) = (50)^2-(7)^2 = 2500-49 = 2451$

b) $45^2 = (40+5)^2 = (40)^2+2(40)(5)+(5)^2 = 1600+400+25 = 2025$

c) Rewrite 481 as $(400+81)$ and 319 as $(400-81)$.

Then $481 \times 319 = (400+81)(400-81) = (400)^2-(81)^2$

$(400)^2$ is easy - it equals 160000.

$(81)^2$ is not easy to do mentally, so let's rewrite 81 as $80+1$.

$(81)^2 = (80+1)^2 = (80)^2+2(80)(1)+(1)^2 = 6400+160+1 = 6561$

Then $481 \times 319 = (400)^2-(81)^2 = 160000-6561 = 153439$

Vocabulary

- **Square of a binomial:** $(a+b)^2 = a^2+2ab+b^2$, *and* $(a-b)^2 = a^2-2ab+b^2$
- **Sum and difference formula:** $(a+b)(a-b) = a^2-b^2$

Guided Practice

1. *Square the binomial and simplify:* $(5x-2y)^2$.

2. *Multiply* $(4x+5y)(4x-5y)$ *and simplify.*

3. *Use the difference of squares and the binomial square formulas to find the product of* 112×88 *without using a calculator.*

Solutions:

1.) $(5x-2y)^2$

If we let $a = 5x$ and $b = 2y$, then

$$\begin{aligned}(5x-2y)^2 &= (5x)^2 - 2(5x)(2y) + (2y)^2 \\ &= 25x^2 - 20xy + 4y^2\end{aligned}$$

2.) Let $a = 4x$ and $b = 5y$, then:

$$\begin{aligned}(4x+5y)(4x-5y) &= (4x)^2 - (5y)^2 \\ &= 16x^2 - 25y^2\end{aligned}$$

3.) The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

Rewrite 112 as $(100+12)$ and 88 as $(100-12)$.

Then

$$\begin{aligned}112 \times 88 &= (100+12)(100-12) \\ &= (100)^2 - (12)^2 \\ &= 10000 - 144 \\ &= 9856\end{aligned}$$

Practice

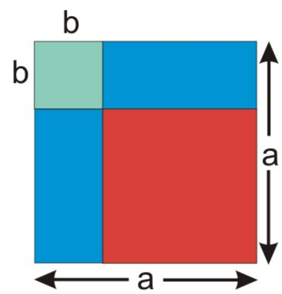
Use the special product rule for squaring binomials to multiply these expressions.

1. $(x+9)^2$
2. $(3x-7)^2$
3. $(5x-y)^2$
4. $(2x^3-3)^2$
5. $(4x^2+y^2)^2$
6. $(8x-3)^2$
7. $(2x+5)(5+2x)$
8. $(xy-y)^2$

Use the special product of a sum and difference to multiply these expressions.

9. $(2x-1)(2x+1)$
10. $(x-12)(x+12)$
11. $(5a-2b)(5a+2b)$
12. $(ab-1)(ab+1)$
13. $(z^2+y)(z^2-y)$
14. $(2q^3+r^2)(2q^3-r^2)$
15. $(7s-t)(t+7s)$
16. $(x^2y+xy^2)(x^2y-xy^2)$

Find the area of the lower right square in the following figure.

17. 

Multiply the following numbers using special products.

18. 45×55
19. 56^2
20. 1002×998
21. 36×44
22. 10.5×9.5
23. 100.2×9.8
24. -95×-105
25. 2×-2

7.5 Monomial Factors of Polynomials

Here you will learn to find a common factor in a polynomial and factor it out of the polynomial.

Can you write the following polynomial as a product of a monomial and a polynomial?

$12x^4+6x^3+3x^2$

Guidance

In the past you have studied common factors of two numbers. Consider the numbers 25 and 35. A common factor of 25 and 35 is 5 because 5 goes into both 25 and 35 evenly.

This idea can be extended to polynomials. A common factor of a polynomial is a number and/or variable that are a factor in all terms of the polynomial. The Greatest Common Factor (or GCF) is the largest monomial that is a factor of each of the terms of the polynomial.

To factor a polynomial means to write the polynomial as a product of other polynomials. One way to factor a polynomial is:

1. Look for the greatest common factor.
2. Write the polynomial as a product of the **greatest common factor** and **the polynomial that results when you divide all the terms of the original polynomial by the greatest common factor.**

One way to think about this type of factoring is that you are essentially doing the distributive property in reverse.

Example A

Factor the following binomial: $5a+15$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5 and 15 can both be divided by 5. The GCF for this binomial is 5.

Step 2: Divide the GCF out of each term of the binomial:

$5a+15=5(a+3)$

Example B

Factor the following polynomial: $4x^2+8x-2$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 4, 8 and 2 can all be divided by 2. The GCF for this polynomial is 2.

Step 2: Divide the GCF out of each term of the polynomial:

$4x^2+8x-2=2(2x^2+4x-1)$

Example C

Factor the following polynomial: $3x^5 - 9x^3 - 6x^2$

Solution: *Step 1:* Identify the GCF. Looking at each of the terms, you can see that 3, 9 and 6 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$3x^5 - 9x^3 - 6x^2 = 3x^2(x^3 - 3x - 2)$

Concept Problem Revisited

To write as a product you want to try to factor the polynomial: $12x^4 + 6x^3 + 3x^2$.

Step 1: Identify the GCF of the polynomial. Looking at each of the numbers, you can see that 12, 6, and 3 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$12x^4 + 6x^3 + 3x^2 = 3x^2(4x^2 + 2x + 1)$$

Vocabulary

Common Factor
: ***Common factors*** are numbers (numerical coefficients) or letters (literal coefficients) that are a factor in all parts of the polynomials.

Greatest Common Factor
: The ***Greatest Common Factor*** (or ***GCF***) is the largest monomial that is a factor of (or divides into evenly) each of the terms of the polynomial.

Guided Practice

1. Find the common factors of the following: $a^2(b+7) - 6(b+7)$
2. Factor the following polynomial: $5k^6 + 15k^4 + 10k^3 + 25k^2$
3. Factor the following polynomial: $27x^3y + 18x^2y^2 + 9xy^3$

Answers:

1. *Step 1:* Identify the GCF

This problem is a little different in that if you look at the expression you notice that $(b+7)$ is common in both terms. Therefore $(b+7)$ is the common factor. The GCF for this expression is $(b+7)$.

Step 2: Divide the GCF out of each term of the expression:

$$a^2(b+7) - 6(b+7) = (a^2 - 6)(b+7)$$

2. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5, 15, 10, and 25 can all be divided by 5. Also notice that each of the terms has an k^2 in common. The GCF for this polynomial is $5k^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$5k^6 + 15k^4 + 10k^3 + 25k^2 = 5k^2(k^4 + 3k^2 + 2k + 5)$$

3. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 27, 18 and 9 can all be divided by 9. Also notice that each of the terms has an xy in common. The GCF for this polynomial is $9xy$.

Step 2: Divide the GCF out of each term of the polynomial:
$27x^3y + 18x^2y^2 + 9xy^3 = 9xy(3x^2 + 2xy + y^2)$

Practice

Factor the following polynomials by looking for a common factor:

1. $7x^2 + 14$
2. $9c^2 + 3$
3. $8a^2 + 4a$
4. $16x^2 + 24y^2$
5. $2x^2 - 12x + 8$
6. $32w^2x + 16xy + 8x^2$
7. $12abc + 6bcd + 24acd$
8. $15x^2y - 10x^2y^2 + 25x^2y$
9. $12a^2b - 18ab^2 - 24a^2b^2$
10. $4s^3t^2 - 16s^2t^3 + 12st^2 - 24st^3$

Find the common factors of the following expressions and then factor:

1. $2x(x-5) + 7(x-5)$
2. $4x(x-3) + 5(x-3)$
3. $3x^2(e+4) - 5(e+4)$
4. $8x^2(c-3) - 7(c-3)$
5. $ax(x-b) + c(x-b)$

7.6 Factoring When the Leading Coefficient Equals 1

Here you'll learn how to factor a quadratic equation in standard form, when $a = 1$.

The area of a rectangle is $x^2 - 3x - 28$. What are the length and width of the rectangle?

Guidance

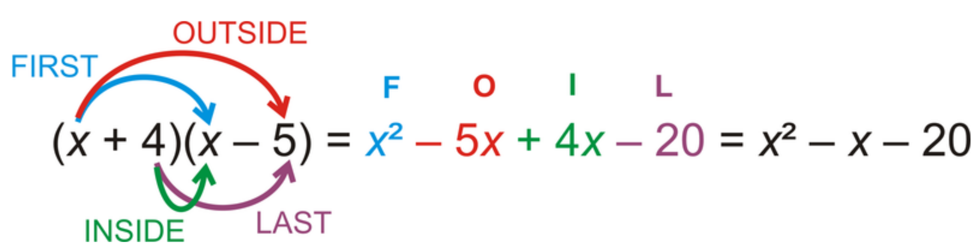
A **quadratic equation** has the form $ax^2 + bx + c$, where $a \neq 0$ (If $a = 0$, then the equation would be linear). For all quadratic equations, the 2 is the largest and only exponent. A quadratic equation can also be called a **trinomial** when all three terms are present.

There are four ways to solve a quadratic equation. The easiest is **factoring**. In this concept, we are going to focus on factoring when $a = 1$ or when there is no number in front of x^2. First, let's start with a review of multiplying two factors together.

Example A

Multiply $(x+4)(x-5)$.

Solution: Even though this is not a quadratic, the product of the two **factors** will be. Remember from previous math classes that a factor is a number that goes evenly into a larger number. For example, 4 and 5 are factors of 20. So, to determine the larger number that $(x+4)$ and $(x-5)$ go into, we need to multiply them together. One method for multiplying two polynomial factors together is called FOIL. To do this, you need to multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and the LAST terms together and then combine like terms.



Therefore $(x+4)(x-5) = x^2 - x - 20$. We can also say that $(x+4)$ and $(x-5)$ are factors of $x^2 - x - 20$.

More Guidance

Now, we will "undo" the multiplication of two factors by factoring. In this concept, we will only address quadratic equations in the form $x^2 + bx + c$, or when $a = 1$.

Investigation: Factoring

1. From the previous example, we know that $(x+m)(x+n) = x^2 + bx + c$.

FOIL $(x+m)(x+n)$.

$$(x+m)(x+n) \Rightarrow x^2 + \underbrace{nx+mx}_{bx} + \underbrace{mn}_{c}$$

2. This shows us that the **constant** term, or c, is equal to the product of the constant numbers inside each factor. It also shows us that the **coefficient** in front of x, or b, is equal to the sum of these numbers.

3. Group together the first two terms and the last two terms. Find the Greatest Common Factor, or GCF, for each pair.

$$\begin{aligned}&(x^2+nx)+(mx+mn)\\&x(x+n)+m(x+n)\end{aligned}$$

4. Notice that what is inside both sets of parenthesis in Step 3 is the same. This number, $(x+n)$, is the GCF of $x(x+n)$ and $m(x+n)$. You can pull it out in front of the two terms and leave the $x+m$.

$$\begin{aligned}&x(x+n)+m(x+n)\\&(x+n)(x+m)\end{aligned}$$

We have now shown how to go from FOIL-ing to factoring and back. Let's apply this idea to an example.

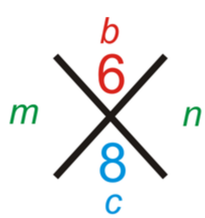
Example B

Factor x^2+6x+8.

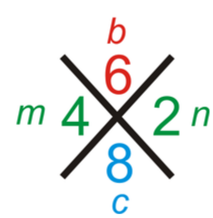
Solution: Let's use the investigation to help us.

$$x^2+6x+8=(x+m)(x+n)$$

So, from Step 2, b will be equal to the sum of m and n and c will be equal to their product. Applying this to our problem, $6=m+n$ and $8=mn$. To organize this, use an "X". Place the sum in the top and the product in the bottom.



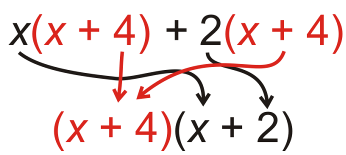
Factors of 8	Sum
4 · 2	6
-4 · -2	-6
1 · 8	9
-1 · -8	-9



The green pair above is the only one that also adds up to 6. Now, move on to Step 3 from our investigation. We need to rewrite the $x-$term, or b, as a sum of m and n.

$$\begin{aligned}&x^2+6x+8\\&\swarrow\searrow\\&x^2+4x+2x+8\\&(x^2+4x)+(2x+8)\\&x(x+4)+2(x+4)\end{aligned}$$

Moving on to Step 4, we notice that the $(x+4)$ term is the same. Pull this out and we are done.



Therefore, the factors of x^2+6x+8 are $(x+4)(x+2)$. You can FOIL this to check your answer.

Example C

Factor $x^2+12x-28$.

Solution: We can approach this problem in exactly the same way we did Example B. This time, we will not use the "X." What are the factors of -28 that also add up to 12? Let's list them out to see:

$$-4\cdot 7, 4\cdot -7, 2\cdot -14, -2\cdot 14, 1\cdot -28, -1\cdot 28$$

The **red** pair above is the one that works. Notice that we only listed the factors of *negative* 28.

$$x^2+12x-28$$

$$x^2-2x+14x-28$$

$$(x^2-2x)+(14x-28)$$

$$x(x-2)+14(x-2)$$

$$(x-2)(x+14)$$

By now, you might have a couple questions:

1. Does it matter which $x-$term you put first? NO, order does not matter. In the previous example, we could have put $14x$ followed by $-2x$. We would still end up with the same answer.
2. Can I skip the "expanded" part (Steps 3 and 4 in the investigation)? YES and NO. Yes, if $a=1$ No, if $a\neq 1$ (the next concept). If $a=1$, then $x^2+bx+c=(x+m)(x+n)$ such that $m+n=b$ and $mn=c$. Consider this a shortcut.

Example D

Factor x^2-4x.

Solution: This is an example of a quadratic that is not a trinomial because it only has two terms, also called a **binomial**. There is no c, or constant term. To factor this, we need to look for the GCF. In this case, the largest number that can be taken out of both terms is an x.

$$x^2-4x=x(x-4)$$

Therefore, the factors are x and $x-4$.

Intro Problem Revisit Recall that the area of a rectangle is $A = lw$, where l is the length and w is the width. To find the length and width, we can therefore factor the area $x^2 - 3x - 28$.

What are the factors of –28 that add up to –3? Testing the various possibilities, we find that $-7 \cdot 4 = -28$ and $-7 + 4 = -3$.

Therefore, $x^2 - 3x - 28$ factors to $(x-7)(x+4)$, and one of these factors is the rectangle's length while the other is its width.

Guided Practice

1. Multiply $(x-3)(x+8)$.

Factor the following quadratics, if possible.

2. $x^2 - 9x + 20$

3. $x^2 + 7x - 30$

4. $x^2 + x + 6$

5. $x^2 + 10x$

Answers

1. FOIL-ing our factors together, we get:

$$(x-3)(x+8) = x^2 + 8x - 3x - 24 = x^2 + 5x - 24$$

2. Using the "X," we have:

From the shortcut above, $-4 + -5 = -9$ and $-4 \cdot -5 = 20$.

$$x^2 - 9x + 20 = (x-4)(x-5)$$

3. Let's list out all the factors of -30 and their sums. The sums are in red.

$$-10 \cdot 3\ (-7), -3 \cdot 10\ (7), -2 \cdot 15\ (13), -15 \cdot 2\ (-13), -1 \cdot 30\ (29), -30 \cdot 1\ (-29)$$

From this, the factors of -30 that add up to 7 are -3 and 10. $x^2 + 7x - 30 = (x-3)(x+10)$

4. There are no factors of 6 that add up to 1. If we had -6, then the trinomial would be factorable. But, as is, this is not a factorable trinomial.

5. The only thing we can do here is to take out the GCF. $x^2 + 10x = x(x+10)$

Vocabulary

Quadratic Equation
An equation where the largest exponent is a 2 and has the form ax^2+bx+c, $a \neq 0$.

Trinomial
A quadratic equation with three terms.

Binomial
A quadratic equation with two terms.

Factoring
A way to break down a quadratic equation into smaller factors.

Factor
A number that goes evenly into a larger number.

FOIL
A method used to multiply together two factors. You multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and LAST terms and then combine any like terms.

Coefficient
The number in front of a variable.

Constant
A number that is added or subtracted within an equation.

Practice

Multiply the following factors together.

1. $(x+2)(x-8)$
2. $(x-9)(x-1)$
3. $(x+7)(x+3)$

Factor the following quadratic equations. If it cannot be factored, write *not factorable*. You can use either method presented in the examples.

4. x^2-x-2
5. $x^2+2x-24$
6. x^2-6x
7. x^2+6x+9
8. $x^2+8x-10$
9. $x^2-11x+30$
10. $x^2+13x-30$
11. $x^2+11x+28$
12. $x^2-8x+12$
13. $x^2-7x-44$
14. $x^2-8x-20$

15. x^2+4x+3
16. $x^2-5x+36$
17. $x^2-5x-36$
18. x^2+x

Challenge Fill in the X's below with the correct numbers.

19. 

20. 

7.7 Factoring When the Leading Coefficient Doesn't Equal 1

Here you'll learn how to factor a quadratic equation in standard form, by expanding the x-term.

The area of a square is $9x^2+24x+16$. What are the dimensions of the square?

Guidance

When we add a number in front of the x^2 term, it makes factoring a little trickier. We still follow the investigation from the previous section, but ***cannot*** use the shortcut. First, let's try FOIL-ing when the coefficients in front of the $x-$terms are not 1.

Example A

Multiply $(3x-5)(2x+1)$

Solution: We can still use FOIL.

FIRST $3x \cdot 2x = 6x^2$

OUTSIDE $3x \cdot 1 = 3x$

INSIDE $-5 \cdot 2x = -10x$

LAST $-5 \cdot 1 = -5$

Combining all the terms together, we get: $6x^2+3x-10x-5 = 6x^2-7x-5$.

Now, let's work backwards and factor a trinomial to get two factors. Remember, you can always check your work by multiplying the final factors together.

Example B

Factor $6x^2-x-2$.

Solution: This is a factorable trinomial. When there is a coefficient, or number in front of x^2, you must follow all the steps from the investigation in the previous concept; no shortcuts. Also, m and n no longer have a product of c and a sum of b. This would not take the coefficient of x^2 into account. What we need to do is multiply together a and c (from ax^2+bx+c) and then find the two numbers whose product is ac and sum is b. Let's use the X to help us organize this.

Now, we can see, we need the two factors of -12 that also add up to -1.

TABLE 7.1:

Factors	Sum
$-1, 12$	11
1, -12	-11
2, -6	-4
-2, 6	4
$3, -4$	-1
-3, 4	1

The factors that work are 3 and -4. Now, take these factors and rewrite the $x-$term expanded using 3 and -4 (Step 3 from the investigation in the previous concept).

$$6x^2 - x - 2$$

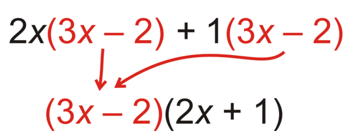
$$\swarrow \searrow$$

$$6x^2 - 4x + 3x - 2$$

Next, group the first two terms together and the last two terms together and pull out any common factors.

$$(6x^2 - 4x) + (3x - 2)$$
$$2x(3x - 2) + 1(3x - 2)$$

Just like in the investigation, what is in the parenthesis is *the same*. We now have two terms that both have $(3x - 2)$ as factor. Pull this factor out.



The factors of $6x^2 - x - 2$ are $(3x - 2)(2x + 1)$. You can FOIL these to check your answer.

Example C

Factor $4x^2 + 8x - 5$.

Solution: Let's make the steps from Example B a little more concise.

1. Find *ac* and the factors of this number that add up to *b*.

$4 \cdot -5 = -20$ The factors of -20 that add up to 8 are 10 and -2.

2. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 1.

$$4x^2 + 8x - 5$$

$$\swarrow \searrow$$

$$4x^2 + 10x - 2x - 5$$

3. Group the first two and second two terms together, find the GCF and factor again.

$$\begin{aligned}&(4x^2+10x)+(-2x-5)\\&2x(2x+5)-1(2x+5)\\&(2x+5)(2x-1)\end{aligned}$$

Alternate Method: What happens if we list $-2x$ before $10x$ in Step 2?

$$\begin{aligned}&4x^2-2x+10x-5\\&(4x^2-2x)(10x-5)\\&2x(2x-1)+5(2x-1)\\&(2x-1)(2x+5)\end{aligned}$$

This tells us it does not matter which $x-$term we list first in Step 2 above.

Example D

Factor $12x^2-22x-20$.

Solution: Let's use the steps from Example C, but we are going to add an additional step at the beginning.

1. Look for any common factors. Pull out the GCF of all three terms, if there is one.

$$12x^2-22x-20=2(6x^2-11x-10)$$

This will make it much easier for you to factor what is inside the parenthesis.

2. Using what is inside the parenthesis, find ac and determine the factors that add up to b.

$$6\cdot -10=-60 \rightarrow -15\cdot 4=-60,\ -15+4=-11$$

The factors of -60 that add up to -11 are -15 and 4.

3. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 2.

$$\begin{aligned}&2(6x^2-11x-10)\\&2(6x^2-15x+4x-10)\end{aligned}$$

4. Group the first two and second two terms together, find the GCF and factor again.

$$\begin{aligned}&2(6x^2-15x+4x-10)\\&2\left[(6x^2-15x)+(4x-10)\right]\\&2\left[3x(2x-5)+2(2x-5)\right]\\&2(2x-5)(3x+2)\end{aligned}$$

Intro Problem Revisit The dimensions of a square are its length and its width, so we need to factor the area $9x^2 + 24x + 16$.

We need to multiply together a and c (from $ax^2 + bx + c$) and then find the two numbers whose product is ac and whose sum is b.

Now we can see that we need the two factors of 144 that also add up to 24. Testing the possibilities, we find that $12 \cdot 12 = 144$ and $12 + 12 = 24$.

Now, take these factors and rewrite the $x-$term expanded using 12 and 12.

$$9x^2 + 24x + 16$$
$$\swarrow \searrow$$
$$9x^2 + 12x + 12x + 16$$

Next, group the first two terms together and the last two terms together and pull out any common factors.

$$(9x^2 + 12x) + (12x + 16)$$
$$3x(3x + 4) + 4(3x + 4)$$

We now have two terms that both have $(3x + 4)$ as factor. Pull this factor out.

The factors of $9x^2 + 24x + 16$ are $(3x + 4)(3x + 4)$, which are also the dimensions of the square.

Guided Practice

1. Multiply $(4x - 3)(3x + 5)$.

Factor the following quadratics, if possible.

2. $15x^2 - 4x - 3$

3. $3x^2 + 6x - 12$

4. $24x^2 - 30x - 9$

5. $4x^2 + 4x - 48$

Answers

1. FOIL: $(4x - 3)(3x + 5) = 12x^2 + 20x - 9x - 15 = 12x^2 + 11x - 15$

2. Use the steps from the examples above. There is no GCF, so we can find the factors of ac that add up to b.

$15 \cdot -3 = -45$ The factors of -45 that add up to -4 are -9 and 5.

$$15x^2 - 4x - 3$$
$$(15x^2 - 9x) + (5x - 3)$$
$$3x(5x - 3) + 1(5x - 3)$$
$$(5x - 3)(3x + 1)$$

3. $3x^2+6x-12$ has a GCF of 3. Pulling this out, we have $3(x^2+2x-6)$. There is no number in front of x^2, so we see if there are any factors of -6 that add up to 2. There are not, so this trinomial is not factorable.

4. $24x^2-30x-9$ also has a GCF of 3. Pulling this out, we have $3(8x^2-10x-3)$. $ac=-24$. The factors of -24 than add up to -10 are -12 and 2.

$$3(8x^2-10x-3)$$
$$3\left[(8x^2-12x)+(2x-3)\right]$$
$$3\left[4x(2x-3)+1(2x-3)\right]$$
$$3(2x-3)(4x+1)$$

5. $4x^2+4x-48$ has a GCF of 4. Pulling this out, we have $4(x^2+x-12)$. This trinomial does not have a number in front of x^2, so we can use the shortcut from the previous concept. What are the factors of -12 that add up to 1?

$$4(x^2+x-12)$$
$$4(x+4)(x-3)$$

Practice

Multiply the following expressions.

1. $(2x-1)(x+5)$
2. $(3x+2)(2x-3)$
3. $(4x+1)(4x-1)$

Factor the following quadratic equations, if possible. If they cannot be factored, write *not factorable*. Don't forget to look for any GCFs first.

4. $5x^2+18x+9$
5. $6x^2-21x$
6. $10x^2-x-3$
7. $3x^2+2x-8$
8. $4x^2+8x+3$
9. $12x^2-12x-18$
10. $16x^2-6x-1$
11. $5x^2-35x+60$
12. $2x^2+7x+3$
13. $3x^2+3x+27$
14. $8x^2-14x-4$
15. $10x^2+27x-9$
16. $4x^2+12x+9$
17. $15x^2+35x$
18. $6x^2-19x+15$
19. Factor x^2-25. What is b?
20. Factor $9x^2-16$. What is b? What types of numbers are a and c?

7.8 Factoring Special Quadratics

Here you'll learn to factor perfect square trinomials and the difference of squares.

The total time, in hours, it takes a rower to paddle upstream, turn around and come back to her starting point is $18x^2 = 32$. How long does it take her to make the round trip?

Guidance

There are a couple of special quadratics that, when factored, have a pattern.

Investigation: Multiplying the Square of a Binomal

1. Rewrite $(a+b)^2$ as the product of two factors. Expand $(a+b)^2$. $(a+b)^2 = (a+b)(a+b)$
2. FOIL your answer from Step 1. This is a **perfect square trinomial.** $a^2+2ab+b^2$
3. $(a-b)^2$ also produces a perfect square trinomial. $(a-b)^2 = a^2-2ab+b^2$
4. Apply the formula above to factoring $9x^2-12x+4$. First, find a and b.

$$\begin{aligned} a^2 &= 9x^2, \ b^2 = 4 \\ a &= 3x, \quad b = 2 \end{aligned}$$

5. Now, plug a and b into the appropriate formula.

$$\begin{aligned} (3x-2)^2 &= (3x)^2 - 2(3x)(2) + 2^2 \\ &= 9x^2 - 12x + 4 \end{aligned}$$

Investigation: Multiplying (a b)(a - b)

1. FOIL $(a-b)(a+b)$.

$$\begin{aligned} (a-b)(a+b) &= a^2 + ab - ab - b^2 \\ &= a^2 - b^2 \end{aligned}$$

2. This is a **difference of squares.** The difference of squares will always factor to be $(a+b)(a-b)$.
3. Apply the formula above to factoring $25x^2-16$. First, find a and b.

$$\begin{aligned} a^2 &= 25x^2, \ b^2 = 16 \\ a &= 5x, \quad b = 4 \end{aligned}$$

4. Now, plug a and b into the appropriate formula. $(5x-4)(5x+4)=(5x)^2-4^2$

**It is important to note that if you forget these formulas or do not want to use them, you can still factor all of these quadratics the same way you did in the previous two concepts.

Example A

Factor x^2-81.

Solution: Using the formula from the investigation above, we need to first find the values of a and b.

$$\begin{aligned} x^2-81 &= a^2-b^2 \\ a^2=x^2&,\ b^2=81 \\ a=x,&\quad b=9 \end{aligned}$$

Now, plugging x and 9 into the formula, we have $x^2-81=(x-9)(x+9)$. To solve for a and b, we found the **square root** of each number. Recall that the square root is a number that, when multiplied by itself, produces another number. This other number is called a **perfect square**.

Alternate Method

Rewrite x^2-81 so that the middle term is present. $x^2+0x-81$

Using the method from the previous two concepts, what are the two factors of -81 that add up to 0? 9 and -9

Therefore, the factors are $(x-9)(x+9)$.

Example B

Factor $36x^2+120x+100$.

Solution: First, check for a GCF.

$$4(9x^2+30x+25)$$

Now, double-check that the quadratic equation above fits into the perfect square trinomial formula.

$$\begin{array}{ccc} a^2=9x^2 & b^2=25 & \\ \sqrt{a^2}=\sqrt{9x^2} & \sqrt{b^2}=\sqrt{25} & 2ab=30x \\ a=3x & b=5 & 2(3x)(5)=30x \end{array}$$

Using a and b above, the equation factors to be $4(3x+5)^2$. If you did not factor out the 4 in the beginning, the formula will still work. a would equal $6x$ and b would equal 10, so the factors would be $(6x+10)^2$. If you expand and find the GCF, you would have $(6x+10)^2=(6x+10)(6x+10)=2(3x+5)2(3x+5)=4(3x+5)^2$.

Alternate Method

First, find the GCF. $4(9x^2+30x+25)$

Then, find ac and expand b accordingly. $9\cdot 25=225$, the factors of 225 that add up to 30 are 15 and 15.

$$4(9x^2+30x+25)$$
$$4(9x^2+15x+15x+25)$$
$$4\left[(9x^2+15x)+(15x+25)\right]$$
$$4\left[3x(3x+5)+5(3x+5)\right]$$
$$4(3x+5)(3x+5) \ or \ 4(3x^2+5)$$

Again, notice that if you do not use the formula discovered in this concept, you can still factor and get the correct answer.

Example C

Factor $48x^2-147$.

Solution: At first glance, this does not look like a difference of squares. 48 nor 147 are square numbers. But, if we take a 3 out of both, we have $3(16x^2-49)$. 16 and 49 are both square numbers, so now we can use the formula.

$$\begin{aligned} 16x^2 &= a^2 & 49 &= b^2 \\ 4x &= a & 7 &= b \end{aligned}$$

The factors are $3(4x-7)(4x+7)$.

Intro Problem Revisit $18x^2=32$ can be rewritten as $18x^2-32=0$, so factor $18x^2-32$.

First, we must take greatest common factor of 2 out of both. We then have $2(9x^2-16)$. 9 and 16 are both square numbers, so now we can use the formula.

$$\begin{aligned} 9x^2 &= a^2 & 16 &= b^2 \\ 3x &= a & 4 &= b \end{aligned}$$

The factors are $2(3x-4)(3x+4)$.

Finally, to find the time, set these factors equal to zero and solve $2(3x-4)(3x+4)=0$.

Because x represents the time, it must be positive. Only $(3x-4)=0$ results in a positive value of x.

$x=\frac{4}{3}=1.3333$ Therefore the round trip takes 1.3333 hours.

You will do more problems like this one in the next lesson.

Guided Practice

Factor the following quadratic equations.

1. x^2-4
2. $2x^2-20x+50$
3. $81x^2+144+64$

Answers

1. $a=x$ and $b=2$. Therefore, $x^2-4=(x-2)(x+2)$.

2. Factor out the GCF, 2. $2(x^2-10x+25)$. This is now a perfect square trinomial with $a=x$ and $b=5$.

$$2(x^2-10x+25)=2(x-5)^2.$$

3. This is a perfect square trinomial and no common factors. Solve for a and b.

$$\begin{aligned} 81x^2 &= a^2 & 64 &= b^2 \\ 9x &= a & 8 &= b \end{aligned}$$

The factors are $(9x+8)^2$.

Vocabulary

Perfect Square Trinomial
A quadratic equation in the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$.

Difference of Squares
A quadratic equation in the form a^2-b^2.

Square Root
A number, that when multiplied by itself produces another number. 3 is the square root of 9.

Perfect Square
A number that has a square root that is an integer. 25 is a perfect square.

Practice

1. List the perfect squares that are less than 200.
2. Why do you think there is no *sum of squares* formula?

Factor the following quadratics, if possible.

3. x^2-1
4. x^2+4x+4
5. $16x^2-24x+9$
6. $-3x^2+36x-108$
7. $144x^2-49$
8. $196x^2+140x+25$
9. $100x^2+1$
10. $162x^2+72x+8$
11. $225-x^2$
12. $121-132x+36x^2$
13. $5x^2+100x-500$

14. $256x^2 - 676$
15. **Error Analysis** Spencer is given the following problem: Multiply $(2x-5)^2$. Here is his work:

$$(2x-5)^2 = (2x)^2 - 5^2 = 4x^2 - 25$$

His teacher tells him the answer is $4x^2 - 20x + 25$. What did Spencer do wrong? Describe his error and correct the problem.

7.9 Zero Product Principle

Here you'll learn how to apply the zero-product property and how to factor polynomials to solve for their unknown variables.

What if you had a polynomial equation like $3x^2+4x-4=0$? How could you factor the polynomial to solve the equation? After completing this Concept, you'll be able to solve polynomial equations by factoring and by using the zero-product property.

Guidance

The most useful thing about factoring is that we can use it to help solve polynomial equations.

Example A

Consider an equation like $2x^2+5x-42=0$. How do you solve for x?

Solution:

There's no good way to isolate x in this equation, so we can't solve it using any of the techniques we've already learned. But the left-hand side of the equation can be factored, making the equation $(x+6)(2x-7)=0$.

How is this helpful? The answer lies in a useful property of multiplication: if two numbers multiply to zero, then at least one of those numbers must be zero. This is called the **Zero-Product Property.**

What does this mean for our polynomial equation? Since the product equals 0, then at least one of the factors on the left-hand side must equal zero. So we can find the two solutions by setting each factor equal to zero and solving each equation separately.

Setting the factors equal to zero gives us:

$$(x+6)=0 \qquad \text{OR} \qquad (2x-7)=0$$

Solving both of those equations gives us:

$$\begin{aligned} x+6&=0 \\ \underline{\underline{x=-6}} \end{aligned} \qquad \text{OR} \qquad \begin{aligned} 2x-7&=0 \\ 2x&=7 \\ \underline{\underline{x=\tfrac{7}{2}}} \end{aligned}$$

Notice that the solution is $x=-6$ **OR** $x=\frac{7}{2}$. The **OR** means that either of these values of x would make the product of the two factors equal to zero. Let's plug the solutions back into the equation and check that this is correct.

$$\begin{aligned}
&Check: x=-6; \\
&(x+6)(2x-7)= \\
&(-6+6)(2(-6)-7)= \\
&(0)(-19)=0
\end{aligned}
\qquad
\begin{aligned}
&Check: x=\frac{7}{2} \\
&(x+6)(2x-7)= \\
&\left(\frac{7}{2}+6\right)\left(2\cdot\frac{7}{2}-7\right)= \\
&\left(\frac{19}{2}\right)(7-7)= \\
&\left(\frac{19}{2}\right)(0)=0
\end{aligned}$$

Both solutions check out.

Factoring a polynomial is very useful because the Zero-Product Property allows us to break up the problem into simpler separate steps. When we can't factor a polynomial, the problem becomes harder and we must use other methods that you will learn later.

As a last note in this section, keep in mind that the Zero-Product Property only works when a product equals zero. For example, if you multiplied two numbers and the answer was nine, that wouldn't mean that one or both of the numbers must be nine. In order to use the property, the factored polynomial must be equal to zero.

Example B

Solve each equation:

a) $(x-9)(3x+4)=0$

b) $x(5x-4)=0$

c) $4x(x+6)(4x-9)=0$

Solution

Since all the polynomials are in factored form, we can just set each factor equal to zero and solve the simpler equations separately

a) $(x-9)(3x+4)=0$ can be split up into two linear equations:

$$\begin{aligned}
&x-9=0 \\
&\underline{\underline{x=9}}
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
&3x+4=0 \\
&3x=-4 \\
&\underline{\underline{x=-\frac{4}{3}}}
\end{aligned}$$

b) $x(5x-4)=0$ can be split up into two linear equations:

$$\underline{\underline{x=0}}
\qquad \text{or} \qquad
\begin{aligned}
&5x-4=0 \\
&5x=4 \\
&\underline{\underline{x=\frac{4}{5}}}
\end{aligned}$$

c) $4x(x+6)(4x-9)=0$ can be split up into three linear equations:

$$4x = 0 \qquad \text{or} \qquad x+6=0 \qquad \text{or} \qquad 4x-9=0$$
$$x = \frac{0}{4} \qquad\qquad x=-6 \qquad\qquad 4x = 9$$
$$\underline{\underline{x = 0}} \qquad\qquad\qquad \underline{\underline{x=\frac{9}{4}}}$$

Solve Simple Polynomial Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-Product Property to solve polynomials in factored form—now we can use that knowledge to solve polynomials by factoring them first. Here are the steps:

a) If necessary, **rewrite** the equation in standard form so that the right-hand side equals zero.

b) **Factor** the polynomial completely.

c) Use the zero-product rule to **set each factor equal to zero.**

d) **Solve** each equation from step 3.

e) **Check** your answers by substituting your solutions into the original equation

Example C

Solve the following polynomial equations.

a) $x^2 - 2x = 0$

b) $2x^2 = 5x$

c) $9x^2y - 6xy = 0$

Solution

a) $x^2 - 2x = 0$

Rewrite: this is not necessary since the equation is in the correct form.

Factor: The common factor is x, so this factors as $x(x-2) = 0$.

Set each factor equal to zero:

$$x = 0 \qquad \text{or} \qquad x-2=0$$

Solve:

$$\underline{x=0} \qquad \text{or} \qquad \underline{x=2}$$

Check: Substitute each solution back into the original equation.

$$x = 0 \Rightarrow (0)^2 - 2(0) = 0 \qquad \text{works out}$$
$$x = 2 \Rightarrow (2)^2 - 2(2) = 4-4=0 \qquad \text{works out}$$

Answer: $x = 0, x = 2$

b) $2x^2 = 5x$

Rewrite: $2x^2 = 5x \Rightarrow 2x^2 - 5x = 0$

Factor: The common factor is x, so this factors as $x(2x-5) = 0$.

Set each factor equal to zero:

$$x = 0 \quad \text{or} \quad 2x - 5 = 0$$

Solve:

$$\underline{x = 0} \quad \text{or} \quad 2x = 5, \quad \underline{x = \frac{5}{2}}$$

Check: Substitute each solution back into the original equation.

$$x = 0 \Rightarrow 2(0)^2 = 5(0) \Rightarrow 0 = 0 \qquad \text{works out}$$
$$x = \frac{5}{2} \Rightarrow 2\left(\frac{5}{2}\right)^2 = 5 \cdot \frac{5}{2} \Rightarrow 2 \cdot \frac{25}{4} = \frac{25}{2} \Rightarrow \frac{25}{2} = \frac{25}{2} \qquad \text{works out}$$

Answer: $x = 0, x = \frac{5}{2}$

c) $9x^2y - 6xy = 0$

Rewrite: not necessary

Factor: The common factor is $3xy$, so this factors as $3xy(3x-2) = 0$.

Set each factor equal to zero:

$3 = 0$ is never true, so this part does not give a solution. The factors we have left give us:

$$x = 0 \quad \text{or} \quad y = 0 \quad \text{or} \quad 3x - 2 = 0$$

Solve:

$$\underline{x = 0} \quad \text{or} \quad \underline{y = 0} \quad \text{or} \quad 3x = 2, \quad \underline{x = \frac{2}{3}}$$

Check: Substitute each solution back into the original equation.

$$x = 0 \Rightarrow 9(0)y - 6(0)y = 0 - 0 = 0 \qquad \text{works out}$$
$$y = 0 \Rightarrow 9x^2(0) - 6x(0) = 0 - 0 = 0 \qquad \text{works out}$$
$$x = \frac{2}{3} \Rightarrow 9 \cdot \left(\frac{2}{3}\right)^2 y - 6 \cdot \frac{2}{3} y = 9 \cdot \frac{4}{9} y - 4y = 4y - 4y = 0 \qquad \text{works out}$$

Answer: $x = 0, y = 0, x = \frac{2}{3}$

Vocabulary

- Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:
- The **factored form** of a polynomial means it is written as a product of its factors.
- **Zero Product Property:** The only way a product is zero is if one or more of the terms are equal to zero:

$a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

Guided Practice

Solve the following polynomial equation.

$$9x^2 - 3x = 0$$

Solution: $9x^2 - 3x = 0$

Rewrite: This is not necessary since the equation is in the correct form.

Factor: The common factor is $3x$, so this factors as: $3x(3x-1) = 0$.

Set each factor equal to zero.

$$3x = 0 \qquad \text{or} \qquad x - 2 = 0$$

Solve:

$$x = 0 \qquad \text{or} \qquad x = 2$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 & (0)^2 - 2(0) = 0 \\ x = 2 & (2)^2 - 2(2) = 0 \end{array}$$

Answer $x = 0, \ x = 2$

Practice

Solve the following polynomial equations.

1. $x(x+12) = 0$
2. $(2x+1)(2x-1) = 0$
3. $(x-5)(2x+7)(3x-4) = 0$
4. $2x(x+9)(7x-20) = 0$

5. $x(3+y)=0$
6. $x(x-2y)=0$
7. $18y-3y^2=0$
8. $9x^2=27x$
9. $4a^2+a=0$
10. $b^2-\frac{5}{3}b=0$
11. $4x^2=36$
12. $x^3-5x^2=0$

7.10 Quadratic Formula

Here you'll learn how to use the quadratic formula to find the vertex and solution of quadratic equations.

What if you had a quadratic equation like x^2+5x+2 that you could not easily factor? How could you use its coeffent values to solve it? After completing this Concept, you'll be able to use the quadratic formula to solve equations like this one.

Guidance

The **Quadratic Formula** is probably the most used method for solving quadratic equations. For a quadratic equation in standard form, $ax^2+bx+c=0$, the quadratic formula looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation: $ax^2+bx+c=0$

Subtract the constant term from both sides: $ax^2+bx=-c$

Divide by the coefficient of the x^2 term:

$x^2+\frac{b}{a}x=-\frac{c}{a}$

Rewrite:

$x^2+2\left(\frac{b}{2a}\right)x=-\frac{c}{a}$

Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides:

$x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$

Factor the perfect square trinomial:

$\left(x+\frac{b}{2a}\right)^2=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}$

Simplify:

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Take the square root of both sides:

$x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{4a^2}}$ and $x+\frac{b}{2a}=-\sqrt{\frac{b^2-4ac}{4a^2}}$

Simplify:

$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a}$ and $x+\frac{b}{2a}=-\frac{\sqrt{b^2-4ac}}{2a}$

$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$ and $x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$

$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$

This can be written more compactly as $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious, so the quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

To use the quadratic formula, just plug in the values of a, b, and c.

Example A

Solve the following quadratic equations using the quadratic formula.

a) $2x^2+3x+1=0$

b) $x^2-6x+5=0$

c) $-4x^2+x+1=0$

Solution

Start with the quadratic formula and plug in the values of a, b and c.

a)

$$\begin{aligned}
\text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
\text{Plug in the values } a=2,\ b=3,\ c=1 \quad & x=\frac{-3\pm\sqrt{(3)^2-4(2)(1)}}{2(2)}\\
\text{Simplify:} \quad & x=\frac{-3\pm\sqrt{9-8}}{4}=\frac{-3\pm\sqrt{1}}{4}\\
\text{Separate the two options:} \quad & x=\frac{-3+1}{4} \text{ and } x=\frac{-3-1}{4}\\
\text{Solve:} \quad & x=\frac{-2}{4}=-\frac{1}{2} \text{ and } x=\frac{-4}{4}=-1
\end{aligned}$$

Answer: $x=-\frac{1}{2}$ and $x=-1$

b)

$$\begin{aligned}
\text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
\text{Plug in the values } a=1,\ b=-6,\ c=5 \quad & x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(5)}}{2(1)}\\
\text{Simplify:} \quad & x=\frac{6\pm\sqrt{36-20}}{2}=\frac{6\pm\sqrt{16}}{2}\\
\text{Separate the two options:} \quad & x=\frac{6+4}{2} \text{ and } x=\frac{6-4}{2}\\
\text{Solve:} \quad & x=\frac{10}{2}=5 \text{ and } x=\frac{2}{2}=1
\end{aligned}$$

Answer: $x=5$ and $x=1$

c)

Quadratic formula:	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=-4,\ b=1,\ c=1$	$x=\frac{-1\pm\sqrt{(1)^2-4(-4)(1)}}{2(-4)}$
Simplify:	$x=\frac{-1\pm\sqrt{1+16}}{-8}=\frac{-1\pm\sqrt{17}}{-8}$
Separate the two options:	$x=\frac{-1+\sqrt{17}}{-8}$ and $x=\frac{-1-\sqrt{17}}{-8}$
Solve:	$x=-.39$ and $x=.64$

Answer: $x=-.39$ and $x=.64$

Often when we plug the values of the coefficients into the quadratic formula, we end up with a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced math classes, you'll learn how to work with "complex" (or "imaginary") solutions to quadratic equations.

Example B

Use the quadratic formula to solve the equation $x^2+2x+7=0$.

Solution

Quadratic formula:	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=1,\ b=2,\ c=7$	$x=\frac{-2\pm\sqrt{(2)^2-4(1)(7)}}{2(1)}$
Simplify:	$x=\frac{-2\pm\sqrt{4-28}}{2}=\frac{-2\pm\sqrt{-24}}{2}$

Answer: There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, that means we have to start by rewriting the equation.

Finding the Vertex of a Parabola with the Quadratic Formula

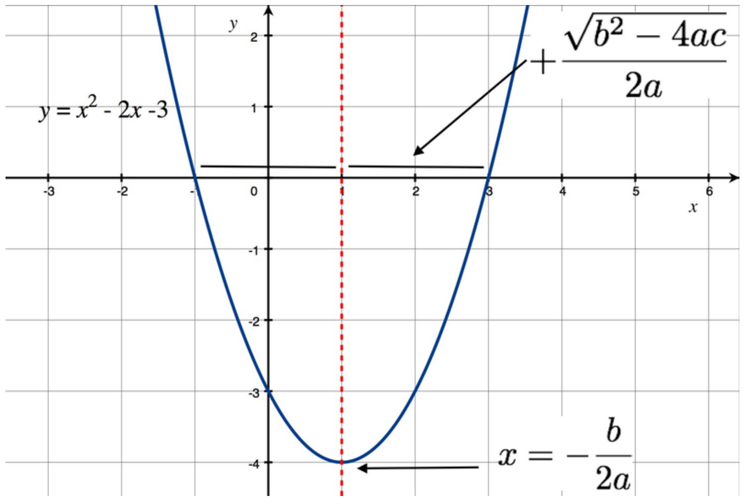
Sometimes a formula gives you even more information than you were looking for. For example, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

Remember that the quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$,, and we can rewrite that as $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$.

Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x=\frac{-b}{2a}$.

Example C

In the equation $x^2-2x-3=0$, the roots -1 and 3 are both 2 units from the vertical line $x=1$, as you can see in the graph below:



Vocabulary

- For a **quadratic equation** in standard form, $ax^2+bx+c=0$, the **quadratic formula** looks like this:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- The quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and we can rewrite that as $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$.
- The roots are symmetric about the **vertex**. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x = \frac{-b}{2a}$.

Guided Practice

Solve the following equations using the quadratic formula.

a) $x^2-6x=10$

b) $-8x^2=5x+6$

Solution

a)

Re-write the equation in standard form:	$x^2-6x-10=0$
Quadratic formula:	$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=1,\ b=-6,\ c=-10$	$x = \frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-10)}}{2(1)}$
Simplify:	$x = \frac{6\pm\sqrt{36+40}}{2} = \frac{6\pm\sqrt{76}}{2}$
Separate the two options:	$x = \frac{6+\sqrt{76}}{2}$ and $x = \frac{6-\sqrt{76}}{2}$
Solve:	$x = 7.36$ and $x = -1.36$

Answer: $x = 7.36$ and $x = -1.36$

b)

Re-write the equation in standard form:	$8x^2 + 5x + 6 = 0$
Quadratic formula:	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Plug in the values $a = 8,\ b = 5,\ c = 6$	$x = \frac{-5 \pm \sqrt{(5)^2 - 4(8)(6)}}{2(8)}$
Simplify:	$x = \frac{-5 \pm \sqrt{25 - 192}}{16} = \frac{-5 \pm \sqrt{-167}}{16}$

Answer: no real solutions

Practice

Solve the following quadratic equations using the quadratic formula.

1. $x^2 + 4x - 21 = 0$
2. $x^2 - 6x = 12$
3. $3x^2 - \frac{1}{2}x = \frac{3}{8}$
4. $2x^2 + x - 3 = 0$
5. $-x^2 - 7x + 12 = 0$
6. $-3x^2 + 5x = 2$
7. $4x^2 = x$
8. $x^2 + 2x + 6 = 0$
9. $5x^2 - 2x + 100 = 0$
10. $100x^2 + 10x + 70 = 0$

7.11 Vertex, Intercept, and Standard Form

Here you'll explore the different forms of the quadratic equation.

The profit on your school fundraiser is represented by the quadratic expression $-5p^2+400p-8000$, where p is your price point. What price point will result in a maximum profit and what is that profit?

Guidance

So far, we have only used the **standard form** of a quadratic equation, $y = ax^2+bx+c$ to graph a parabola. From standard form, we can find the vertex and either factor or use the Quadratic Formula to find the $x-$intercepts. The **intercept form** of a quadratic equation is $y = a(x-p)(x-q)$, where a is the same value as in standard form, and p and q are the $x-$intercepts. This form looks very similar to a factored quadratic equation.

Example A

Change $y = 2x^2+9x+10$ to intercept form and find the vertex. Graph the parabola.

Solution: First, let's change this equation into intercept form by factoring. $ac = 20$ and the factors of 20 that add up to 9 are 4 and 5. Expand the $x-$term.

$$\begin{aligned}
y &= 2x^2+9x+10\\
y &= 2x^2+4x+5x+10\\
y &= 2x(x+2)+5(x+2)\\
y &= (2x+5)(x+2)
\end{aligned}$$

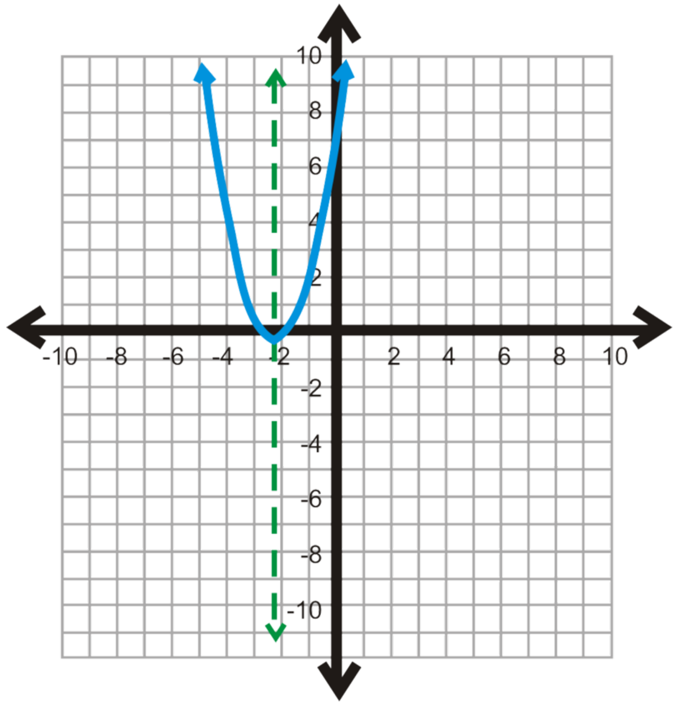
Notice, this does not exactly look like the definition. The factors cannot have a number in front of x. Pull out the 2 from the first factor to get $y = 2\left(x+\frac{5}{2}\right)(x+2)$. Now, find the vertex. Recall that all parabolas are symmetrical. This means that the axis of symmetry is *halfway* between the $x-$intercepts or their average.

$$\text{axis of symmetry} = \frac{p+q}{2} = \frac{-\frac{5}{2}-2}{2} = -\frac{9}{2} \div 2 = -\frac{9}{2}\cdot\frac{1}{2} = -\frac{9}{4}$$

This is also the $x-$coordinate of the vertex. To find the $y-$coordinate, plug the $x-$value into either form of the quadratic equation. We will use Intercept form.

$$\begin{aligned}
y &= 2\left(-\frac{9}{4}+\frac{5}{2}\right)\left(-\frac{9}{4}+2\right)\\
y &= 2\cdot\frac{1}{4}\cdot-\frac{1}{4}\\
y &= -\frac{1}{8}
\end{aligned}$$

The vertex is $\left(-2\frac{1}{4}, -\frac{1}{8}\right)$. Plot the $x-$intercepts and the vertex to graph.

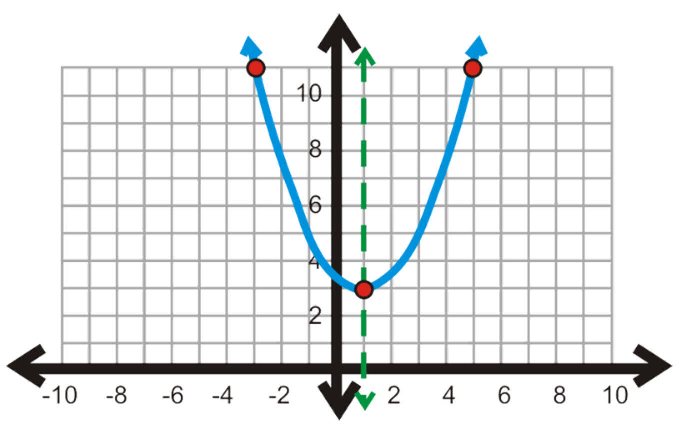


The last form is vertex form. **Vertex form** is written $y = a(x-h)^2 + k$, where (h,k) is the vertex and a is the same is in the other two forms. Notice that h is negative in the equation, but positive when written in coordinates of the vertex.

Example B

Find the vertex of $y = \frac{1}{2}(x-1)^2 + 3$ and graph the parabola.

Solution: The vertex is going to be (1, 3). To graph this parabola, use the symmetric properties of the function. Pick a value on the left side of the vertex. If $x = -3$, then $y = \frac{1}{2}(-3-1)^2 + 3 = 11$. -3 is 4 units away from 1 (the $x-$coordinate of the vertex). 4 units on the *other* side of 1 is 5. Therefore, the $y-$coordinate will be 11. Plot (1, 3), (-3, 11), and (5, 11) to graph the parabola.



Example C

Change $y = x^2 - 10x + 16$ into vertex form.

Solution: To change an equation from standard form into vertex form, you must complete the square. Review the *Completing the Square* Lesson if needed. The major difference is that you will not need to solve this equation.

$$\begin{aligned} y &= x^2-10x+16 \\ y-16+25 &= x^2-10x+25 && \text{Move 16 to the other side and add } \left(\frac{b}{2}\right)^2 \text{ to both sides.} \\ y+9 &= (x-5)^2 && \text{Simplify left side and factor the right side} \\ y &= (x-5)^2-9 && \text{Subtract 9 from both sides to get } y \text{ by itself.} \end{aligned}$$

To solve an equation in vertex form, set $y=0$ and solve for x.

$$\begin{aligned} (x-5)^2-9 &= 0 \\ (x-5)^2 &= 9 \\ x-5 &= \pm 3 \\ x &= 5 \pm 3 \ or \ 8 \ and \ 2 \end{aligned}$$

Intro Problem Revisit The vertex will give us the price point that will result in the maximum profit and that profit, so let's change this equation into intercept form by factoring. First factor out –5.

$$\begin{aligned} -5p^2+400p-8000 &= -5(p^2-80p+1600) \\ &\quad -5(p-40)(p-40) \end{aligned}$$

From this we can see that the x-intercepts are 40 and 40. The average of 40 and 40 is 40 we plug 40 into our original equation.

$$-5(40)^2+400(40)-8000 = -8000+16000-8000 = 0$$

Therefore, the price point that results in a maximum profit is \$40 and that price point results in a profit of \$0. You're not making any money, so you better rethink your fundraising approach!

Guided Practice

1. Find the intercepts of $y=2(x-7)(x+2)$ and change it to standard form.
2. Find the vertex of $y=-\frac{1}{2}(x+4)^2-5$ and change it to standard form.
3. Change $y=x^2+18x+45$ to intercept form and graph.
4. Change $y=x^2-6x-7$ to vertex form and graph.

Answers

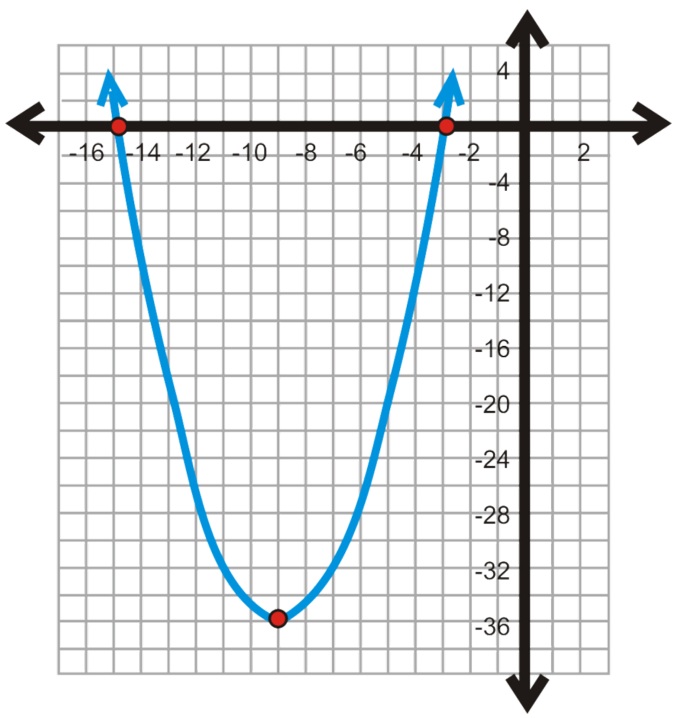
1. The intercepts are the opposite sign from the factors; (7, 0) and (-2, 0). To change the equation into standard form, FOIL the factors and distribute a.

$$y = 2(x-7)(x+2)$$
$$y = 2(x^2-5x-14)$$
$$y = 2x^2-10x-28$$

2. The vertex is (-4, -5). To change the equation into standard form, FOIL $(x+4)^2$, distribute a, and then subtract 5.

$$y = -\frac{1}{2}(x+4)(x+4)-5$$
$$y = -\frac{1}{2}(x^2+8x+16)-5$$
$$y = -\frac{1}{2}x^2-4x-21$$

3. To change $y = x^2+18x+45$ into intercept form, factor the equation. The factors of 45 that add up to 18 are 15 and 3. Intercept form would be $y = (x+15)(x+3)$. The intercepts are (-15, 0) and (-3, 0). The $x-$coordinate of the vertex is halfway between -15 and -3, or -9. The $y-$coordinate of the vertex is $y = (-9)^2+18(-9)+45 = -36$. Here is the graph:

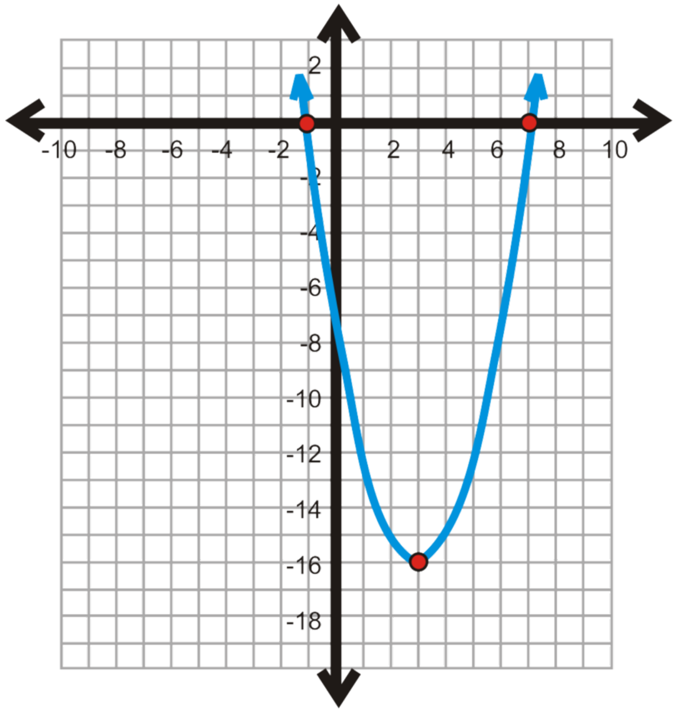


4. To change $y = x^2-6x-7$ into vertex form, complete the square.

$$y+7+9 = x^2-6x+9$$
$$y+16 = (x-3)^2$$
$$y = (x-3)^2-16$$

The vertex is (3, -16).

For vertex form, we could solve the equation by using square roots or we could factor the standard form. Either way, we will get that the $x-$intercepts are (7, 0) and (-1, 0).



Vocabulary

Standard form
: $y = ax^2 + bx + c$

Intercept form
: $y = a(x-p)(x-q)$, where p and q are the $x-$intercepts.

Vertex form
: $y = a(x-h)^2 + k$, where (h,k) is the vertex.

Practice

1. Fill in the table below. Either describe how to find each entry or use a formula.

	Equation	Vertex	Intercepts
Standard Form			
Intercept Form			
Vertex Form			

Find the vertex and $x-$intercepts of each function below. Then, graph the function. If a function does not have any $x-$intercepts, use the symmetry property of parabolas to find points on the graph.

2. $y = (x-4)^2 - 9$

3. $y = (x+6)(x-8)$
4. $y = x^2 + 2x - 8$
5. $y = -(x-5)(x+7)$
6. $y = 2(x+1)^2 - 3$
7. $y = 3(x-2)^2 + 4$
8. $y = \frac{1}{3}(x-9)(x+3)$
9. $y = -(x+2)^2 + 7$
10. $y = 4x^2 - 13x - 12$

Change the following equations to intercept form.

11. $y = x^2 - 3x + 2$
12. $y = -x^2 - 10x + 24$
13. $y = 4x^2 + 18x + 8$

Change the following equations to vertex form.

14. $y = x^2 + 12x - 28$
15. $y = -x^2 - 10x + 24$
16. $y = 2x^2 - 8x + 15$

Change the following equations to standard form.

17. $y = (x-3)^2 + 8$
18. $y = 2\left(x - \frac{3}{2}\right)(x-4)$
19. $y = -\frac{1}{2}(x+6)^2 - 11$